

## ON THE PETERSON HIT PROBLEM

NGUYỄN SUM

ABSTRACT. We study the *hit problem*, set up by F. Peterson, of finding a minimal set of generators for the polynomial algebra  $P_k := \mathbb{F}_2[x_1, x_2, \dots, x_k]$  as a module over the mod-2 Steenrod algebra,  $\mathcal{A}$ . In this paper, we study a minimal set of generators for  $\mathcal{A}$ -module  $P_k$  in some so-called generic degrees and apply these results to explicitly determine the hit problem for  $k = 4$ .

*Dedicated to Prof. N. H. V. Hưng on the occasion of his sixtieth birthday*

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $V_k$  be an elementary abelian 2-group of rank  $k$ . Denote by  $BV_k$  the classifying space of  $V_k$ . It may be thought of as the product of  $k$  copies of the real projective space  $\mathbb{R}P^\infty$ . Then

$$P_k := H^*(BV_k) \cong \mathbb{F}_2[x_1, x_2, \dots, x_k],$$

a polynomial algebra in  $k$  variables  $x_1, x_2, \dots, x_k$ , each of degree 1. Here the cohomology is taken with coefficients in the prime field  $\mathbb{F}_2$  of two elements.

Being the cohomology of a space,  $P_k$  is a module over the mod-2 Steenrod algebra  $\mathcal{A}$ . The action of  $\mathcal{A}$  on  $P_k$  is explicitly given by the formula

$$Sq^i(x_j) = \begin{cases} x_j, & i = 0, \\ x_j^2, & i = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and subject to the Cartan formula

$$Sq^n(fg) = \sum_{i=0}^n Sq^i(f)Sq^{n-i}(g),$$

for  $f, g \in P_k$  (see Steenrod and Epstein [30]).

A polynomial  $f$  in  $P_k$  is called *hit* if it can be written as a finite sum  $f = \sum_{i>0} Sq^i(f_i)$  for some polynomials  $f_i$ . That means  $f$  belongs to  $\mathcal{A}^+P_k$ , where  $\mathcal{A}^+$  denotes the augmentation ideal in  $\mathcal{A}$ . We are interested in the *hit problem*, set up by F. Peterson, of finding a minimal set of generators for the polynomial algebra  $P_k$  as a module over the Steenrod algebra. In other words, we want to find a basis of the  $\mathbb{F}_2$ -vector space  $QP_k := P_k/\mathcal{A}^+P_k = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ .

The hit problem was first studied by Peterson [22, 23], Wood [38], Singer [28], and Priddy [24], who showed its relation to several classical problems respectively in cobordism theory, modular representation theory, Adams spectral sequence for the stable homotopy of spheres, and stable homotopy type of classifying spaces of

<sup>1</sup> 2010 *Mathematics Subject Classification*. Primary 55S10; 55S05, 55T15.

<sup>2</sup> *Keywords and phrases*. Steenrod squares, polynomial algebra, Peterson hit problem.

<sup>3</sup> This version is a revision of a preprint of Quy Nhơn University, Việt Nam, 2011.

finite groups. The vector space  $QP_k$  was explicitly calculated by Peterson [22] for  $k = 1, 2$ , by Kameko [14] for  $k = 3$ . The case  $k = 4$  has been treated by Kameko [16] and by the present author [31].

Several aspects of the hit problem were then investigated by many authors. (See Boardman [1], Bruner, Hà and Hưng [2], Carlisle and Wood [3], Crabb and Hubbuck [4], Giambalvo and Peterson [5], Hà [6], Hưng [7], Hưng and Nam [8, 9], Hưng and Peterson [10, 11], Janfada and Wood [12, 13], Kameko [14, 15], Minami [17], Mothebe [18, 19], Nam [20, 21], Repka and Selick [25], Silverman [26], Silverman and Singer [27], Singer [29], Walker and Wood [35, 36, 37], Wood [39, 40] and others.)

The  $\mu$ -function is one of the numerical functions that have much been used in the context of the hit problem. For a positive integer  $n$ , by  $\mu(n)$  one means the smallest number  $r$  for which it is possible to write  $n = \sum_{1 \leq i \leq r} (2^{d_i} - 1)$ , where  $d_i > 0$ . A routine computation shows that  $\mu(n) = s$  if and only if there exists uniquely a sequence of integers  $d_1 > d_2 > \dots > d_{s-1} \geq d_s > 0$  such that

$$n = 2^{d_1} + 2^{d_2} + \dots + 2^{d_{s-1}} + 2^{d_s} - s. \quad (1.1)$$

From this it implies  $n - s$  is even and  $\mu(\frac{n-s}{2}) \leq s$ .

Denote by  $(P_k)_n$  the subspace of  $P_k$  consisting of all the homogeneous polynomials of degree  $n$  in  $P_k$  and by  $(QP_k)_n$  the subspace of  $QP_k$  consisting of all the classes represented by the elements in  $(P_k)_n$ .

Peterson [22] made the following conjecture, which was subsequently proved by Wood [38].

**Theorem 1.1** (Wood [38]). *If  $\mu(n) > k$ , then  $(QP_k)_n = 0$ .*

One of the main tools in the study of the hit problem is Kameko's homomorphism  $\widetilde{Sq}_*^0 : QP_k \rightarrow QP_k$ . This homomorphism is induced by the  $\mathbb{F}_2$ -linear map, also denoted by  $\widetilde{Sq}_*^0 : P_k \rightarrow P_k$ , given by

$$\widetilde{Sq}_*^0(x) = \begin{cases} y, & \text{if } x = x_1 x_2 \dots x_k y^2, \\ 0, & \text{otherwise,} \end{cases}$$

for any monomial  $x \in P_k$ . Note that  $\widetilde{Sq}_*^0$  is not an  $\mathcal{A}$ -homomorphism. However,  $\widetilde{Sq}_*^0 Sq^{2t} = Sq^t \widetilde{Sq}_*^0$  and  $\widetilde{Sq}_*^0 Sq^{2t+1} = 0$  for any non-negative integer  $t$ .

**Theorem 1.2** (Kameko [14]). *Let  $m$  be a positive integer. If  $\mu(2m + k) = k$ , then  $(\widetilde{Sq}_*^0)_m : (QP_k)_{2m+k} \rightarrow (QP_k)_m$  is an isomorphism of the  $\mathbb{F}_2$ -vector spaces.*

Based on Theorems 1.1 and 1.2, the hit problem is reduced to the case of degree  $n$  with  $\mu(n) = s < k$ .

The hit problem in the case of degree  $n$  of the form (1.1) with  $s = k - 1$ ,  $d_{i-1} - d_i > 1$  for  $2 \leq i < k$  and  $d_{k-1} > 1$  was partially studied by Crabb and Hubbuck [4], Nam [20], Repka and Selick [25] and the present author [33].

In this paper, we explicitly determine the hit problem for the case  $k = 4$ . First, we study the hit problem for the case of degree  $n$  of the form (1.1) for  $s = k - 1$ . The following theorem gives an inductive formula for the dimension of  $(QP_k)_n$  in this case.

**Theorem 1.3.** *Let  $n = \sum_{1 \leq i \leq k-1} (2^{d_i} - 1)$  with  $d_i$  positive integers such that  $d_1 > d_2 > \dots > d_{k-2} \geq d_{k-1}$ , and let  $m = \sum_{1 \leq i \leq k-2} (2^{d_i - d_{k-1}} - 1)$ . If  $d_{k-1} \geq k-1 \geq 3$ , then*

$$\dim(QP_k)_n = (2^k - 1) \dim(QP_{k-1})_m.$$

For  $d_{k-2} > d_{k-1} \geq k$ , the theorem follows from a result in Nam [20]. For  $d_{k-2} = d_{k-1} > k$ , it has been proved in [33]. However, for either  $d_{k-1} = k-1$  or  $d_{k-2} = d_{k-1} = k$ , the theorem is new.

From the results in Peterson [22] and Kameko [14], we see that if  $k = 3$ , then this theorem is true for either  $d_1 > d_2 \geq 2$  or  $d_1 = d_2 \geq 3$ ; if  $k = 2$ , then it is true for  $d_1 \geq 2$ .

The main tool in the proof of the theorem is Singer's criterion on the hit monomials (Theorem 2.12.) So, the condition  $d_1 > d_2 > \dots > d_{k-2} \geq d_{k-1} > 0$  is used in our proof when we use this criterion.

Based on Theorem 1.3, we explicitly compute  $QP_4$ .

**Theorem 1.4.** *Let  $n$  be an arbitrary positive integer with  $\mu(n) < 4$ . The dimension of the  $\mathbb{F}_2$ -vector space  $(QP_4)_n$  is given by the following table:*

$n$	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s \geq 5$
$2^{s+1} - 3$	4	15	35	45	45
$2^{s+1} - 2$	6	24	50	70	80
$2^{s+1} - 1$	14	35	75	89	85
$2^{s+2} + 2^{s+1} - 3$	46	94	105	105	105
$2^{s+3} + 2^{s+1} - 3$	87	135	150	150	150
$2^{s+4} + 2^{s+1} - 3$	136	180	195	195	195
$2^{s+t+1} + 2^{s+1} - 3, t \geq 4$	150	195	210	210	210
$2^{s+1} + 2^s - 2$	21	70	116	164	175
$2^{s+2} + 2^s - 2$	55	126	192	240	255
$2^{s+3} + 2^s - 2$	73	165	241	285	300
$2^{s+4} + 2^s - 2$	95	179	255	300	315
$2^{s+5} + 2^s - 2$	115	175	255	300	315
$2^{s+t} + 2^s - 2, t \geq 6$	125	175	255	300	315
$2^{s+2} + 2^{s+1} + 2^s - 3$	64	120	120	120	120
$2^{s+3} + 2^{s+2} + 2^s - 3$	155	210	210	210	210
$2^{s+t+1} + 2^{s+t} + 2^s - 3, t \geq 3$	140	210	210	210	210
$2^{s+3} + 2^{s+1} + 2^s - 3$	140	225	225	225	225
$2^{s+u+1} + 2^{s+1} + 2^s - 3, u \geq 3$	120	210	210	210	210
$2^{s+u+2} + 2^{s+2} + 2^s - 3, u \geq 2$	225	315	315	315	315
$2^{s+t+u} + 2^{s+t} + 2^s - 3, u \geq 2, t \geq 3$	210	315	315	315	315

The vector space  $QP_4$  was also computed in Kameko [16] by using computer calculation. However the manuscript is unpublished at the time of the writing.

Carlisle and Wood showed in [3] that the dimension of the vector space  $(QP_k)_n$  is uniformly bounded by a number depended only on  $k$ . In 1990, Kameko made the following conjecture in his Johns Hopkins University PhD thesis [14].

**Conjecture 1.5** (Kameko [14]). *For every non-negative integer  $n$ ,*

$$\dim(QP_k)_n \leq \prod_{1 \leq i \leq k} (2^i - 1).$$

The conjecture was shown by Kameko himself for  $k \leq 3$  in [14]. From Theorem 1.4, we see that the conjecture is also true for  $k = 4$ .

By induction on  $k$ , using Theorem 1.3, we obtain the following.

**Corollary 1.6.** *Let  $n = \sum_{1 \leq i \leq k-1} (2^{d_i} - 1)$  with  $d_i$  positive integers. If  $d_1 - d_2 \geq 2$ ,  $d_{i-1} - d_i \geq i - 1$ ,  $3 \leq i \leq k - 1$ ,  $d_{k-1} \geq k - 1 \geq 2$ , then*

$$\dim(QP_k)_n = \prod_{1 \leq i \leq k} (2^i - 1).$$

For the case  $d_{i-1} - d_i \geq i$ ,  $2 \leq i \leq k - 1$ , and  $d_{k-1} \geq k$ , this result is due to Nam [20]. This corollary also shows that Kameko's conjecture is true for the degree  $n$  as given in the corollary.

By induction on  $k$ , using Theorems 1.3, 1.4 and the fact that Kameko's homomorphism is an epimorphism, one gets the following.

**Corollary 1.7.** *Let  $n = \sum_{1 \leq i \leq k-2} (2^{d_i} - 1)$  with  $d_i$  positive integers and let  $d_{k-1} = 1$ ,  $n_r = \sum_{1 \leq i \leq r-2} (2^{d_i - d_{r-1}} - 1) - 1$  with  $r = 5, 6, \dots, k$ . If  $d_1 - d_2 \geq 4$ ,  $d_{i-2} - d_{i-1} \geq i$ , for  $4 \leq i \leq k$  and  $k \geq 5$ , then*

$$\dim(QP_k)_n = \prod_{1 \leq i \leq k} (2^i - 1) + \sum_{5 \leq r \leq k} \left( \prod_{r+1 \leq i \leq k} (2^i - 1) \right) \dim \text{Ker}(\widetilde{Sq}_*^0)_{n_r},$$

where  $(\widetilde{Sq}_*^0)_{n_r} : (QP_r)_{2n_r+r} \rightarrow (QP_r)_{n_r}$  denotes Kameko's homomorphism  $\widetilde{Sq}_*^0$  in degree  $2n_r + r$ . Here, by convention,  $\prod_{r+1 \leq i \leq k} (2^i - 1) = 1$  for  $r = k$ .

This corollary has been proved in [33] for the case  $d_{i-2} - d_{i-1} > i + 1$  with  $3 \leq i \leq k$ .

Obviously  $2n_r + r = \sum_{1 \leq i \leq r-2} (2^{e_i} - 1)$ , where  $e_i = d_i - d_{r-1} + 1$ , for  $1 \leq i \leq r - 2$ . So, in degree  $2n_r + r$  of  $P_r$ , there is a so-called spike  $x = x_1^{2^{e_1}-1} x_2^{2^{e_2}-1} \dots x_{r-2}^{2^{e_{r-2}}-1}$ , i.e. a monomial whose exponents are all of the form  $2^e - 1$  for some  $e$ . Since the class  $[x]$  in  $(QP_k)_{2n_r+r}$  represented by the spike  $x$  is nonzero and  $\widetilde{Sq}_*^0([x]) = 0$ , we have  $\text{Ker}(\widetilde{Sq}_*^0)_{n_r} \neq 0$ , for any  $5 \leq r \leq k$ . Therefore, by Corollary 1.7, Kameko's conjecture is not true in degree  $n = 2n_k + k$  for any  $k \geq 5$ , where  $n_k = 2^{d_1-1} + 2^{d_2-1} + \dots + 2^{d_{k-2}-1} - k + 1$ .

This paper is organized as follows. In Section 2, we recall some needed information on the admissible monomials in  $P_k$  and Singer's criterion on the hit monomials. We prove Theorem 1.3 in Section 3 by describing a basis of  $(QP_k)_n$  in terms of a given basis of  $(QP_{k-1})_m$ . In Section 4, we recall the results on the hit problem for  $k \leq 3$ . Theorem 1.4 will be proved in Section 5 by explicitly determining all of the admissible monomials in  $P_4$ .

The first formulation of this paper was given in a 240-page preprint in 2007 [31], which was then publicized to a remarkable number of colleagues. One year latter, we found the negative answer to Kameko's conjecture on the hit problem [32, 33]. Being led by the insight of this new study, we have remarkably reduced the length of the paper.

The main results of the present paper have already been announced in [34]. However, we correct Theorem 3 in [34] by replacing the condition  $d_{k-1} \geq k-1 \geq 1$  with  $d_{k-1} \geq k-1 \geq 3$ .

## 2. PRELIMINARIES

In this section, we recall some results in Kameko [14] and Singer [29] which will be used in the next sections.

**Notation 2.1.** Throughout the paper, we use the following notations.

$$\begin{aligned}\mathbb{N}_k &= \{1, 2, \dots, k\}, \\ X_{\mathbb{J}} &= X_{\{j_1, j_2, \dots, j_s\}} = \prod_{j \in \mathbb{N}_k \setminus \mathbb{J}} x_j, \quad \mathbb{J} = \{j_1, j_2, \dots, j_s\} \subset \mathbb{N}_k,\end{aligned}$$

In particular, we have

$$\begin{aligned}X_{\mathbb{N}_k} &= 1, \\ X_{\emptyset} &= x_1 x_2 \dots x_k, \\ X_j &= X_{\{j\}} = x_1 \dots \hat{x}_j \dots x_k, \quad 1 \leq j \leq k.\end{aligned}$$

Let  $\alpha_i(a)$  denote the  $i$ -th coefficient in dyadic expansion of a non-negative integer  $a$ . That means  $a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \dots$ , for  $\alpha_i(a) = 0$  or  $1$  and  $i \geq 0$ . Denote by  $\alpha(a)$  the number of 1's in dyadic expansion of  $a$ .

Let  $x = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} \in P_k$ . Denote  $\nu_j(x) = a_j, 1 \leq j \leq k$ . Set

$$\mathbb{J}_i(x) = \{j \in \mathbb{N}_k : \alpha_i(\nu_j(x)) = 0\},$$

for  $i \geq 0$ . Then we have

$$x = \prod_{i \geq 0} X_{\mathbb{J}_i(x)}^{2^i}.$$

For a polynomial  $f$  in  $P_k$ , we denote by  $[f]$  the class in  $QP_k$  represented by  $f$ . For a subset  $S \subset P_k$ , we denote

$$[S] = \{[f] : f \in S\} \subset QP_k.$$

**Definition 2.2.** For a monomial  $x$ , define two sequences associated with  $x$  by

$$\begin{aligned}\omega(x) &= (\omega_1(x), \omega_2(x), \dots, \omega_i(x), \dots), \\ \sigma(x) &= (\nu_1(x), \nu_2(x), \dots, \nu_k(x)),\end{aligned}$$

where  $\omega_i(x) = \sum_{1 \leq j \leq k} \alpha_{i-1}(\nu_j(x)) = \deg X_{I_{i-1}(x)}, i \geq 1$ .

The sequence  $\omega(x)$  is called the weight vector of  $x$  (see Wood [39]). The weight vectors and the sigma vectors can be ordered by the left lexicographical order.

Let  $\omega = (\omega_1, \omega_2, \dots, \omega_i, \dots)$  be a sequence of non-negative integers such that  $\omega_i = 0$  for  $i \gg 0$ . Define  $\deg \omega = \sum_{i \geq 0} 2^{i-1} \omega_i$ . Denote by  $P_k(\omega)$  the subspace of  $P_k$  spanned by all monomials  $y$  such that  $\deg y = \deg \omega$ ,  $\omega(y) \leq \omega$ , and by  $P_k^-(\omega)$  the subspace of  $P_k$  spanned by all monomials  $y \in P_k(\omega)$  such that  $\omega(y) < \omega$ . Denote by  $\mathcal{A}_s^+$  the subspace of  $\mathcal{A}$  spanned by all  $Sq^j$  with  $1 \leq j < 2^s$ .

**Definition 2.3.** Let  $\omega$  be a sequence of non-negative integers and  $f, g$  two polynomials of the same degree in  $P_k$ .

- i)  $f \equiv g$  if and only if  $f - g \in \mathcal{A}^+ P_k$ .
- ii)  $f \simeq_{(s, \omega)} g$  if and only if  $f - g \in \mathcal{A}_s^+ P_k + P_k^-(\omega)$ .

Since  $\mathcal{A}_0^+ P_k = 0$ ,  $f \simeq_{(0, \omega)} g$  if and only if  $f - g \in P_k^-(\omega)$ . If  $x$  is a monomial in  $P_k$  and  $\omega = \omega(x)$ , then we denote  $x \simeq_s g$  if and only if  $x \simeq_{(s, \omega(x))} g$ .

Obviously, the relations  $\equiv$  and  $\simeq_{(s, \omega)}$  are equivalence relations.

We recall some relations on the action of the Steenrod squares on  $P_k$ .

**Proposition 2.4.** Let  $f$  be a polynomial in  $P_k$ .

- i) If  $i > \deg f$ , then  $Sq^i(f) = 0$ . If  $i = \deg f$ , then  $Sq^i(f) = f^2$ .
- ii) If  $i$  is not divisible by  $2^s$ , then  $Sq^i(f^{2^s}) = 0$  while  $Sq^{r2^s}(f^{2^s}) = (Sq^r(f))^{2^s}$ .

**Proposition 2.5.** Let  $x, y$  be monomials and let  $f, g$  be polynomials in  $P_k$  such that  $\deg x = \deg f$ ,  $\deg y = \deg g$ .

- i) If  $\omega_i(x) \leq 1$  for  $i > s$  and  $x \simeq_t f$  with  $t \leq s$ , then  $xy^{2^s} \simeq_t fy^{2^s}$ .
- ii) If  $\omega_i(x) = 0$  for  $i > s$ ,  $x \simeq_s f$  and  $y \simeq_r g$ , then  $xy^{2^s} \simeq_{s+r} fg^{2^s}$ .

*Proof.* Suppose that

$$x + f + \sum_{1 \leq u < 2^t} Sq^u(z_u) = h \in P_k^-(\omega(x)), \quad (2.1)$$

where  $z_u \in P_k$ . From this and Proposition 2.4, we have  $Sq^u(z_u)y^{2^s} = Sq^u(z_u y^{2^s})$  for  $1 \leq u < 2^t \leq 2^s$ . Observe that  $\omega_v(xy^{2^s}) = \omega_v(x)$  for  $1 \leq v \leq s$ . If  $z$  is a monomial and  $z \in P_k^-(\omega(x))$ , then there exists an index  $i \geq 1$  such that  $\omega_j(z) = \omega_j(x)$  for  $j \leq i-1$  and  $\omega_i(z) < \omega_i(x)$ . If  $i > s$ , then  $\omega_i(x) = 1, \omega_i(z) = 0$ . Then we have

$$\alpha_{i-1} \left( \deg x - \sum_{1 \leq j \leq i-1} 2^{j-1} \omega_j(x) \right) = \alpha_{i-1} \left( 2^{i-1} + \sum_{j > i} 2^{j-1} \omega_j(x) \right) = 1.$$

On the other hand, since  $\deg x = \deg z$ ,  $\omega_i(z) = 0$  and  $\omega_j(z) = \omega_j(x)$ , for  $j \leq i-1$ , one gets

$$\begin{aligned} \alpha_{i-1} \left( \deg x - \sum_{1 \leq j \leq i-1} 2^{j-1} \omega_j(x) \right) &= \alpha_{i-1} \left( \deg z - \sum_{1 \leq j \leq i-1} 2^{j-1} \omega_j(z) \right) \\ &= \alpha_{i-1} \left( \sum_{j > i} 2^{j-1} \omega_j(z) \right) = 0. \end{aligned}$$

This is a contradiction. Hence,  $1 \leq i \leq s$ .

From the above equalities and the fact that  $h \in P_k^-(\omega(x))$ , one gets

$$xy^{2^s} + fy^{2^s} + \sum_{1 \leq i < 2^t} Sq^i(z_i y^{2^s}) = hy^{2^s} \in P_k^-(\omega(xy^{2^s})).$$

The first part of the proposition is proved.

Suppose that  $y + g + \sum_{1 \leq j < 2^r} Sq^j(u_j) = h_1 \in P_k^-(\omega(y))$ , where  $u_j \in P_k$ . Then

$$xy^{2^s} = xg^{2^s} + xh_1^{2^s} + \sum_{1 \leq j < 2^r} xSq^{j2^s}(u_j^{2^s}).$$

Since  $\omega_i(x) = 0$  for  $i > s$  and  $h_1 \in P_k^-(\omega(y))$ , we get  $xh_1^{2^s} \in P_k^-(\omega(xy^{2^s}))$ . Using the Cartan formula and Proposition 2.4, we obtain

$$xSq^{j2^s}(u_j^{2^s}) = Sq^{j2^s}(xu_j^{2^s}) + \sum_{0 < b \leq j} Sq^{b2^s}(x)(Sq^{j-b}(u_j))^{2^s}.$$

Since  $\omega_i(x) = 0$  for  $i > s$ , we have  $x = \prod_{0 \leq i < s} X_{\mathbb{J}_i(x)}^{2^i}$ . Using the Cartan formula and Proposition 2.4, we see that  $Sq^{b2^s}(x)$  is a sum of polynomials of the form

$$\prod_{0 \leq i < s} (Sq^{b_i}(X_{\mathbb{J}_i(x)}))^{2^i},$$

where  $\sum_{0 \leq i < s} b_i 2^i = b2^s$  and  $0 \leq b_i \leq \deg X_{\mathbb{J}_i(x)}$ . Let  $\ell$  be the smallest index such that  $b_\ell > 0$  with  $0 \leq \ell < s$ . Suppose that a monomial  $z$  appears as a term of the polynomial  $\left(\prod_{0 \leq i < s} (Sq^{b_i}(X_{\mathbb{J}_i(x)}))^{2^i}\right)(Sq^{j-b}(u_j))^{2^s}$ . Then  $\omega_u(z) = \deg X_{\mathbb{J}_{u-1}(x)} = \omega_u(x) = \omega_u(xy^{2^s})$  for  $u \leq \ell$ , and  $\omega_{\ell+1}(z) = \deg X_{\mathbb{J}_\ell(x)} - b_\ell < \deg X_{\mathbb{J}_\ell(x)} = \omega_{\ell+1}(x) = \omega_{\ell+1}(xy^{2^s})$ . Hence,

$$\left(\prod_{0 \leq i < s} (Sq^{b_i}(X_{\mathbb{J}_i(x)}))^{2^i}\right)(Sq^{j-b}(u_j))^{2^s} \in P_k^-(\omega(xy^{2^s})).$$

This implies  $Sq^{b2^s}(x)(Sq^{j-b}(u_j))^{2^s} \in P_k^-(\omega(xy^{2^s}))$  for  $0 < b \leq j$ . So, one gets

$$xy^{2^s} + xg^{2^s} + \sum_{1 \leq j < 2^r} Sq^{j2^s}(xu_j^{2^s}) \in P_k^-(\omega(xy^{2^s})).$$

Since  $1 \leq j2^s < 2^{r+s}$  for  $1 \leq j < 2^r$ , we obtain  $xy^{2^s} \simeq_{r+s} xg^{2^s}$ .

Since  $\omega_i(x) = 0$  for  $i > s$  and  $h \in P_k^-(\omega(x))$ , we have  $hg^{2^s} \in P_k^-(\omega(xy^{2^s}))$ . Using Proposition 2.4, the Cartan formula and the relation (2.1) with  $t = s$ , we get

$$xg^{2^s} + fg^{2^s} + \sum_{1 \leq u < 2^s} Sq^u(z_u g^{2^s}) = hg^{2^s} \in P_k^-(\omega(xy^{2^s})).$$

Combining the above equalities gives  $xy^{2^s} + fg^{2^s} \in \mathcal{A}_{r+s}P_k + P_k^-(\omega(xy^{2^s}))$ . This implies  $xy^{2^s} \simeq_{r+s} fg^{2^s}$ . The proposition follows.  $\square$

**Definition 2.6.** Let  $x, y$  be monomials of the same degree in  $P_k$ . We say that  $x < y$  if and only if one of the following holds:

- i)  $\omega(x) < \omega(y)$ ;
- ii)  $\omega(x) = \omega(y)$  and  $\sigma(x) < \sigma(y)$ .

**Definition 2.7.** A monomial  $x$  is said to be inadmissible if there exist monomials  $y_1, y_2, \dots, y_t$  such that  $y_j < x$  for  $j = 1, 2, \dots, t$  and  $x - \sum_{j=1}^t y_j \in \mathcal{A}^+P_k$ .

A monomial  $x$  is said to be admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree  $n$  in  $P_k$  is a minimal set of  $\mathcal{A}$ -generators for  $P_k$  in degree  $n$ .

**Definition 2.8.** A monomial  $x$  is said to be strictly inadmissible if and only if there exist monomials  $y_1, y_2, \dots, y_t$  such that  $y_j < x$ , for  $j = 1, 2, \dots, t$  and  $x - \sum_{j=1}^t y_j \in \mathcal{A}_s^+P_k$  with  $s = \max\{i : \omega_i(x) > 0\}$ .

It is easy to see that if  $x$  is strictly inadmissible, then it is inadmissible. The following theorem is a modification of a result in [14].

**Theorem 2.9** (Kameko [14], Sum [33]). *Let  $x, y, w$  be monomials in  $P_k$  such that  $\omega_i(x) = 0$  for  $i > r > 0$ ,  $\omega_s(w) \neq 0$  and  $\omega_i(w) = 0$  for  $i > s > 0$ .*

- i) *If  $w$  is inadmissible, then  $xw^{2^r}$  is also inadmissible.*
- ii) *If  $w$  is strictly inadmissible, then  $xw^{2^r}y^{2^{r+s}}$  is inadmissible.*

**Proposition 2.10** ([33]). *Let  $x$  be an admissible monomial in  $P_k$ . Then we have*

- i) *If there is an index  $i_0$  such that  $\omega_{i_0}(x) = 0$ , then  $\omega_i(x) = 0$  for all  $i > i_0$ .*
- ii) *If there is an index  $i_0$  such that  $\omega_{i_0}(x) < k$ , then  $\omega_i(x) < k$  for all  $i > i_0$ .*

Now, we recall a result in [29] on the hit monomials in  $P_k$ .

**Definition 2.11.** A monomial  $z$  in  $P_k$  is called a spike if  $\nu_j(z) = 2^{s_j} - 1$  for  $s_j$  a non-negative integer and  $j = 1, 2, \dots, k$ . If  $z$  is a spike with  $s_1 > s_2 > \dots > s_{r-1} \geq s_r > 0$  and  $s_j = 0$  for  $j > r$ , then it is called a minimal spike.

The following is a criterion for the hit monomials in  $P_k$ .

**Theorem 2.12** (Singer [29]). *Suppose  $x \in P_k$  is a monomial of degree  $n$ , where  $\mu(n) \leq k$ . Let  $z$  be the minimal spike of degree  $n$ . If  $\omega(x) < \omega(z)$ , then  $x$  is hit.*

From this theorem, we see that if  $z$  is a minimal spike, then  $P_k^-(\omega(z)) \subset \mathcal{A}^+ P_k$ .

The following lemma has been proved in [33].

**Lemma 2.13** ([33]). *Let  $n = \sum_{1 \leq i \leq k-1} (2^{d_i} - 1)$  with  $d_i$  positive integers such that  $d_1 > d_2 > \dots > d_{k-2} \geq d_{k-1} > 0$ , and let  $x$  be a monomial of degree  $n$  in  $P_k$ . If  $[x] \neq 0$ , then  $\omega_i(x) = k - 1$  for  $1 \leq i \leq d_{k-1}$ .*

The following is a modification of a result in [33].

**Lemma 2.14.** *Let  $n$  be as in Lemma 2.13 and let  $\omega = (\omega_1, \omega_2, \dots)$  be a sequence of non-negative integers such that  $\omega_i = k - 1$ , for  $1 \leq i \leq s \leq d_{k-1}$ ,  $\omega_i \leq 1$  for  $i > s$ ,  $\omega_i = 0$  for  $i \gg 0$ , and  $\deg \omega < n$ . Suppose  $f, g, h, p \in P_k$  with  $\deg f = \deg g = \deg \omega$ ,  $\deg h = \deg p = (n - \deg \omega)/2^s = \sum_{i=1}^{k-1} (2^{d_i-s} - 1) - \sum_{j \geq 1} 2^{j-1} \omega_{s+j}$ .*

- i) *If  $f \simeq_{(s, \omega)} g$ , then  $fh^{2^s} \equiv gh^{2^s}$ .*
- ii) *If  $\omega_i = 0$  for  $i > s$ , and  $h \equiv p$ , then  $fh^{2^s} \equiv fp^{2^s}$ .*

This lemma can easily be proved by using Proposition 2.5, Theorem 2.12 and Lemma 2.13.

For latter use, we set

$$\begin{aligned} P_k^0 &= \langle \{x = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} : a_1 a_2 \dots a_k = 0\} \rangle, \\ P_k^+ &= \langle \{x = x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} : a_1 a_2 \dots a_k > 0\} \rangle. \end{aligned}$$

It is easy to see that  $P_k^0$  and  $P_k^+$  are the  $\mathcal{A}$ -submodules of  $P_k$ . Furthermore, we have the following.

**Proposition 2.15.** *We have a direct summand decomposition of the  $\mathbb{F}_2$ -vector spaces*

$$QP_k = QP_k^0 \oplus QP_k^+.$$

Here  $QP_k^0 = P_k^0 / \mathcal{A}^+ P_k^0$  and  $QP_k^+ = P_k^+ / \mathcal{A}^+ P_k^+$ .

### 3. PROOF OF THEOREM 1.3

We denote

$$\mathcal{N}_k = \{(i; I) : I = (i_1, i_2, \dots, i_r), 1 \leq i < i_1 < \dots < i_r \leq k, 0 \leq r < k\}.$$

Let  $(i; I) \in \mathcal{N}_k$  and  $j \in \mathbb{N}_k$ . Denote by  $r = \ell(I)$  the length of  $I$ , and

$$I \cup j = \begin{cases} I, & \text{if } j \in I, \\ (j, i_1, \dots, i_r), & \text{if } 0 < j < i_1, \\ (i_1, \dots, i_{t-1}, j, i_t, \dots, i_r), & \text{if } i_{t-1} < j < i_t, 2 \leq t \leq r+1. \end{cases}$$



Here  $i_{r+1} = k + 1$ . For  $1 \leq i \leq k$ , define the homomorphism  $f_i = f_{k;i} : P_{k-1} \rightarrow P_k$  of algebras by substituting

$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leq j < i, \\ x_{j+1}, & \text{if } i \leq j < k. \end{cases}$$

**Definition 3.1.** Let  $(i; I) \in \mathcal{N}_k$ , let  $r = \ell(I)$ , and let  $u$  be an integer with  $1 \leq u \leq r$ . A monomial  $x \in P_{k-1}$  is said to be  $u$ -compatible with  $(i; I)$  if all of the following hold:

- i)  $\nu_{i_1-1}(x) = \nu_{i_2-1}(x) = \dots = \nu_{i_{(u-1)}-1}(x) = 2^r - 1$ ,
- ii)  $\nu_{i_u-1}(x) > 2^r - 1$ ,
- iii)  $\alpha_{r-t}(\nu_{i_u-1}(x)) = 1, \forall t, 1 \leq t \leq u$ ,
- iv)  $\alpha_{r-t}(\nu_{i_t-1}(x)) = 1, \forall t, u < t \leq r$ .

Clearly, a monomial  $x \in P_k$  can be  $u$ -compatible with a given  $(i; I) \in \mathcal{N}_k$ ,  $r = \ell(I) > 0$ , for at most one value of  $u$ . By convention,  $x$  is 1-compatible with  $(i; \emptyset)$ .

**Definition 3.2.** Let  $(i; I) \in \mathcal{N}_k$ . Denote  $x_{(I,u)} = x_{i_u}^{2^{r-1}+\dots+2^{r-u}} \prod_{u < t \leq r} x_{i_t}^{2^{r-t}}$  for  $1 \leq u \leq r = \ell(I)$ , and  $x_{(\emptyset,1)} = 1$ . For a monomial  $x$  in  $P_{k-1}$ , we define the monomial  $\phi_{(i;I)}(x)$  in  $P_k$  by setting

$$\phi_{(i;I)}(x) = \begin{cases} (x_i^{2^r-1} f_i(x)) / x_{(I,u)}, & \text{if there exists } u \text{ such that} \\ & x \text{ is } u\text{-compatible with } (i, I), \\ 0, & \text{otherwise.} \end{cases}$$

Then we have an  $\mathbb{F}_2$ -linear map  $\phi_{(i;I)} : P_{k-1} \rightarrow P_k$ . In particular,  $\phi_{(i;\emptyset)} = f_i$ .

For example, let  $I = (j)$  and  $1 \leq i < j \leq k$ . A monomial  $x \in P_{k-1}$  is 1-compatible with  $(i; I)$  if and only if  $\alpha_0(\nu_{j-1}(x)) = 1$  and  $\phi_{(i;I)}(x) = (x_i f_i(x)) / x_j$ .

Let  $k = 4$  and  $I = (2, 3, 4)$ . The monomial  $x = x_1^{12} x_2^6 x_3^9$  is 1-compatible with  $(1; I)$ ;  $y = x_1^7 x_2^{15} x_3^7$  is 2-compatible with  $(1; I)$ ;  $z = x_1^7 x_2^7 x_3^{15}$  is 3-compatible with  $(1; I)$  and  $\phi_{(1;I)}(x) = x_1^7 x_2^8 x_3^4 x_4^8$ ,  $\phi_{(1;I)}(y) = x_1^7 x_2^7 x_3^9 x_4^6$ ,  $\phi_{(1;I)}(z) = x_1^7 x_2^7 x_3^7 x_4^8$ .

Let  $x = X^{2^d-1} y^{2^d}$ , with  $y$  a monomial in  $P_{k-1}$  and  $X = x_1 x_2 \dots, x_{k-1} \in P_{k-1}$ .

If  $r < d$ , then  $x$  is 1-compatible with  $(i; I)$  and

$$\phi_{(i;I)}(x) = \phi_{(i;I)}(X^{2^d-1}) f_i(y)^{2^d} = x_i^{2^r-1} \prod_{1 \leq t \leq r} x_{i_t}^{2^d-2^{r-t}-1} X_{i, i_1, \dots, i_r}^{2^d-1} f_i(y)^{2^d}. \quad (3.1)$$

If  $d = r$ ,  $\nu_{j-1}(y) = 0, j = i_1, i_2, \dots, i_{u-1}$  and  $\nu_{i_u-1}(y) > 0$ , then  $x$  is  $u$ -compatible with  $(i; I)$  and

$$\phi_{(i;I)}(x) = \phi_{(i_u; J_u)}(X^{2^d-1}) f_i(y)^{2^d}, \quad (3.2)$$

where  $J_u = (i_{u+1}, \dots, i_r)$ .

Let  $B$  be a finite subset of  $P_{k-1}$  consisting of some polynomials in degree  $n$ . We set

$$\begin{aligned} \Phi^0(B) &= \bigcup_{1 \leq i \leq k} \phi_{(i;\emptyset)}(B) = \bigcup_{1 \leq i \leq k} f_i(B). \\ \Phi^+(B) &= \bigcup_{(i;I) \in \mathcal{N}_k, 0 < \ell(I) \leq k-1} \phi_{(i;I)}(B) \setminus P_k^0. \\ \Phi(B) &= \Phi^0(B) \bigcup \Phi^+(B). \end{aligned}$$

It is easy to see that if  $B_{k-1}(n)$  is a minimal set of generators for  $\mathcal{A}$ -module  $P_{k-1}$  in degree  $n$ , then  $\Phi^0(B_{k-1}(n))$  is a minimal set of generators for  $\mathcal{A}$ -module  $P_k^0$  in degree  $n$  and  $\Phi^+(B_{k-1}(n)) \subset P_k^+$ .

**Proposition 3.3.** *Let  $n = \sum_{1 \leq i \leq k-1} (2^{d_i} - 1)$  with  $d_i$  positive integers such that  $d_1 > d_2 > \dots > d_{k-2} \geq d_{k-1} \geq k-1 \geq 3$ . If  $B_{k-1}(n)$  is a minimal set of generators for  $\mathcal{A}$ -module  $P_{k-1}$  in degree  $n$ , then  $B_k(n) = \Phi(B_{k-1}(n))$  is also a minimal set of generators for  $\mathcal{A}$ -module  $P_k$  in degree  $n$ .*

For  $d_{k-1} \geq k$ , this proposition is a modification of a result in Nam [20]. For  $d_{k-2} = d_{k-1} > k$ , it has been proved in [33].

The proposition will be proved in Subsection 3.1. We need some lemmas.

**Lemma 3.4.** *Let  $j_0, j_1, \dots, j_{d-1} \in \mathbb{N}_k$ , with  $d$  a positive integer and let  $i = \min\{j_0, \dots, j_{d-1}\}$ ,  $I = (i_1, \dots, i_r)$  with  $\{i_1, \dots, i_r\} = \{j_0, \dots, j_{d-1}\} \setminus \{i\}$ . Then,*

$$x := \prod_{0 \leq t < d} X_{j_t}^{2^t} \simeq_{d-1} \phi_{(i;I)}(X^{2^d-1}).$$

**Lemma 3.5.** *Let  $n = \sum_{1 \leq i \leq k-1} (2^{d_i} - 1)$  with  $d_i$  positive integers such that  $d_1 > d_2 > \dots > d_{k-2} \geq d_{k-1} = d > 0$ ,  $m = \sum_{1 \leq i \leq k-2} (2^{d_i-d} - 1)$ , and let  $y_0$  be a monomial in  $(P_k)_{m-1}$ ,  $y_j = y_0 x_j$  for  $1 \leq j \leq k$ , and  $(i; I) \in \mathcal{N}_k$ .*

i) *If  $r = \ell(I) < d$ , then*

$$\phi_{(i;I)}(X^{2^d-1}) y_i^{2^d} \equiv \sum_{1 \leq j < i} \phi_{(j;I)}(X^{2^d-1}) y_j^{2^d} + \sum_{i < j \leq k} \phi_{(t_j; I^{(j)})}(X^{2^d-1}) y_j^{2^d},$$

where  $t_j = \min(j, I)$ , and  $I^{(j)} = (I \cup j) \setminus \{t_j\}$  for  $j > i$ .

ii) *If  $r + 1 < d$ , then*

$$\phi_{(i;I)}(X^{2^d-1}) y_i^{2^d} \equiv \sum_{1 \leq j < i} \phi_{(j; I \cup i)}(X^{2^d-1}) y_j^{2^d} + \sum_{i < j \leq k} \phi_{(i; I \cup j)}(X^{2^d-1}) y_j^{2^d}.$$

Denote  $I_t = (t+1, t+2, \dots, k)$  for  $1 \leq t \leq k$ . Set

$$Y_t = \sum_{u=t}^k \phi_{(t; I_t)}(X^{2^d-1}) x_u^{2^d}, \quad d \geq k - t + 1.$$

**Lemma 3.6.** *For  $1 \leq t \leq k$ ,*

$$Y_t \simeq_{(k-t+1, \omega)} \sum_{(j; J)} \phi_{(j; J)}(X^{2^d-1}) x_j^{2^d},$$

where the sum runs over all  $(j; J) \in \mathcal{N}_k$  with  $1 \leq j < t$ ,  $J \subset I_{t-1}$ ,  $J \neq I_{t-1}$  and  $\omega = \omega(X_1^{2^d-1} x_1^{2^d})$ .

We assume that all elements of  $B_{k-1}(n)$  are monomials. Denote  $\mathcal{B} = B_{k-1}(n)$ . We set

$$\begin{aligned} \mathcal{C} &= \{z \in \mathcal{B} : \nu_1(z) > 2^{k-1} - 1\}, \\ \mathcal{D} &= \{z \in \mathcal{B} : \nu_1(z) = 2^{k-1} - 1, \nu_2(z) > 2^{k-1} - 1\}, \\ \mathcal{E} &= \{z \in \mathcal{B} : \nu_1(z) = \nu_2(z) = 2^{k-1} - 1\}. \end{aligned}$$

Since  $\omega_k(z) \geq k-3$  for all  $z \in \mathcal{B}$ , we have  $\mathcal{B} = \mathcal{C} \cup \mathcal{D} \cup \mathcal{E}$ . If  $d = d_{k-1} > k-1$ , then  $\mathcal{D} = \mathcal{E} = \emptyset$ . If  $d_{k-2} > d_{k-1} = k-1$ , then  $\mathcal{E} = \emptyset$ .

We set  $\bar{\mathcal{B}} = \{\bar{z} \in P_{k-1} : X^{2^d-1}\bar{z}^{2^d} \in \mathcal{B}\}$ . If either  $d \geq k$  or  $I \neq I_1$ , then  $\phi_{(i;I)}(z) = \phi_{(i;I)}(X^{2^d-1}f_i(\bar{z})^{2^d})$ . If  $d = d_{k-1} = k-1$ , then using the relation (3.2), we have

$$\phi_{(1;I_1)}(z) = \begin{cases} \phi_{(2;I_2)}(X^{2^d-1}f_1(\bar{z})^{2^d}), & \text{if } z \in \mathcal{C}, \\ \phi_{(3;I_3)}(X^{2^d-1}f_2(\bar{z})^{2^d}), & \text{if } z \in \mathcal{D}, \\ \phi_{(4;I_4)}(X^{2^d-1}f_3(\bar{z})^{2^d}), & \text{if } z \in \mathcal{E}. \end{cases} \quad (3.3)$$

For any  $(i; I) \in \mathcal{N}_k$ , we define the homomorphism  $p_{(i;I)} : P_k \rightarrow P_{k-1}$  of algebras by substituting

$$p_{(i;I)}(x_j) = \begin{cases} x_j, & \text{if } 1 \leq j < i, \\ \sum_{s \in I} x_{s-1}, & \text{if } j = i, \\ x_{j-1}, & \text{if } i < j \leq k. \end{cases}$$

Then  $p_{(i;I)}$  is a homomorphism of  $\mathcal{A}$ -modules. In particular, for  $I = \emptyset$ , we have  $p_{(i;\emptyset)}(x_i) = 0$ .

**Lemma 3.7.** *Let  $z \in \mathcal{B}$  and  $(i; I), (j; J) \in \mathcal{N}_k$  with  $\ell(J) \leq \ell(I)$ .*

i) *If either  $d \geq k$  or  $d = k-1$  and  $I \neq I_1$ , then*

$$p_{(j;J)}(\phi_{(i;I)}(z)) \equiv \begin{cases} z, & \text{if } (j; J) = (i; I), \\ 0, & \text{if } (j; J) \neq (i; I). \end{cases}$$

ii) *If  $z \in \mathcal{C}$  and  $d = k-1$ , then*

$$p_{(i;I)}(\phi_{(1;I_1)}(z)) \equiv \begin{cases} z, & \text{if } (i; I) = (1; I_1), \\ 0 \bmod \langle \mathcal{D} \cup \mathcal{E} \rangle, & \text{if } (i; I) = (2; I_2), \\ 0, & \text{otherwise.} \end{cases}$$

iii) *If  $z \in \mathcal{D}$  and  $d = k-1$ , then*

$$p_{(i;I)}(\phi_{(1;I_1)}(z)) \equiv \begin{cases} z, & \text{if } (i; I) = (1; I_1), (1; I_2), (2; I_2), \\ 0 \bmod \langle \mathcal{E} \rangle, & \text{if } (i; I) = (3; I_3), \\ 0, & \text{otherwise.} \end{cases}$$

iv) *If  $z \in \mathcal{E}$  and  $d = k-1$ , then*

$$p_{(i;I)}(\phi_{(1;I_1)}(z)) \equiv \begin{cases} z & \text{if } I_3 \subset I, \\ 0, & \text{otherwise.} \end{cases}$$

The above lemmas will be proved in Subsections 3.2-3.4. In particular, for  $d > k$ , the first part of Lemma 3.7 has been proved in [33].

**Lemma 3.8** (Nam [20]). *Let  $x$  be a monomial in  $P_k$ . Then  $x \equiv \sum \bar{x}$ , where  $\bar{x}$  are monomials with  $\nu_1(\bar{x}) = 2^t - 1$  and  $t = \alpha(\nu_1(x))$ .*

Now, based on Proposition 3.3 and Lemma 3.7, we prove Theorem 1.3.

*Proof of Theorem 1.3.* Denote by  $|S|$  the cardinal of a set  $S$ . It is easy to check that  $|\mathcal{N}_k| = 2^k - 1$ . Let  $(i; I), (j; J) \in \mathcal{N}_k$  with  $\ell(J) \leq \ell(I)$  and  $y, z \in B_{k-1}(n)$ . Suppose that  $\phi_{(j;J)}(y) = \phi_{(i;I)}(z)$ . Using Lemma 3.7, we have  $y \equiv p_{(j;J)}(\phi_{(i;I)}(z)) \not\equiv 0$ . From this, Lemma 3.7 and the relation (3.3), we get  $y = z$  and  $(i; I) = (j; J)$ . Hence,

$$\phi_{(i;I)}(B_{k-1}(n)) \cap \phi_{(j;J)}(B_{k-1}(n)) = \emptyset.$$

for  $(i; I) \neq (j; J)$  and  $|\phi_{(i; I)}(B_{k-1}(n))| = |B_{k-1}(n)|$ . From Proposition 3.3, we have

$$\begin{aligned} \dim(QP_k)_n &= |B_k(n)| = \sum_{(i; I) \in \mathcal{N}_k} |B_{k-1}(n)| \\ &= |\mathcal{N}_k| \dim(QP_{k-1})_n \\ &= (2^k - 1) \dim(QP_{k-1})_n. \end{aligned}$$

Set  $h_u = 2^{d_1-u} + \dots + 2^{d_{k-2}-u} + 2^{d_{k-1}-u} - k + 1$ , for  $0 \leq u \leq d$ . We have  $h_0 = n$ ,  $h_d = m$ ,  $2h_u + k - 1 = h_{u-1}$  and  $\mu(2h_u + k - 1) = k - 1$  for  $1 \leq u \leq d$ . By Theorem 1.2, Kameko's homomorphism  $(\widetilde{Sq}_*)^0_{h_u} : (QP_{k-1})_{h_{u-1}} \rightarrow (QP_{k-1})_{h_u}$  is an isomorphism of the  $\mathbb{F}_2$ -vector spaces. So, the iterated homomorphism

$$(\widetilde{Sq}_*)^d = (\widetilde{Sq}_*)^0_{h_d} \dots (\widetilde{Sq}_*)^0_{h_1} : (QP_{k-1})_n \rightarrow (QP_{k-1})_m$$

is an isomorphism of the  $\mathbb{F}_2$ -vector spaces. Hence,  $\dim(QP_{k-1})_n = \dim(QP_{k-1})_m$ . The theorem is proved.  $\square$

In the remaining part of the section, we prove Proposition 3.3 and Lemmas 3.4-3.7.

### 3.1. Proof of Proposition 3.3.

Denote by  $\mathcal{P}_k(n)$  the subspace of  $(P_k)_n$  spanned by all elements of the set  $B_k(n)$ . Let  $x$  be a monomial of degree  $n$  in  $P_k$  and  $[x] \neq 0$ . By Lemma 2.13, we have  $\omega_i(x) = k - 1$  for  $1 \leq i \leq d_{k-1} = d$ . Hence, we obtain  $x = \left( \prod_{0 \leq t < d} X_{j_t}^{2^t} \right) \bar{y}^{2^d}$ , for suitable monomial  $\bar{y} \in (P_k)_m$ , with  $m = \sum_{1 \leq i \leq k-2} (2^{d_i-d} - 1)$ .

According to Lemmas 3.4 and 2.14, there is  $(i; I) \in \mathcal{N}_k$  such that

$$x = \left( \prod_{0 \leq t < d} X_{j_t}^{2^t} \right) \bar{y}^{2^d} \equiv \phi_{(i; I)}(X^{2^d-1}) \bar{y}^{2^d},$$

where  $r = \ell(I) < d$ .

Now, we prove  $[x] \in [\mathcal{P}_k(n)]$ . The proof is divided into many cases.

**Case 3.1.1.** If  $\bar{y} = f_i(y)$  with  $y \in (P_{k-1})_m$ , then  $[x] \in [\mathcal{P}_k(n)]$ .

Since the iterated homomorphism  $(\widetilde{Sq}_*)^d : (QP_{k-1})_n \rightarrow (QP_{k-1})_m$  is an isomorphism of the  $\mathbb{F}_2$ -vector spaces,

$$\bar{\mathcal{B}} = (\widetilde{Sq}_*)^d(B_{k-1}(n)) = \{\bar{z} \in (P_{k-1})_m : X^{2^d-1} \bar{z}^{2^d} \in B_{k-1}(n)\}$$

is a minimal set of  $\mathcal{A}$ -generators for  $P_{k-1}$  in degree  $m$ .

Since  $y \in (P_{k-1})_m$ , we have  $y \equiv \bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_s$  with  $\bar{z}_t$  monomials in  $\bar{\mathcal{B}}$ . Using Lemma 2.14, we have

$$x \equiv \phi_{(i; I)}(X^{2^d-1}) f_i(y)^{2^d} \equiv \sum_{1 \leq t \leq s} \phi_{(i; I)}(X^{2^d-1}) f_i(\bar{z}_t)^{2^d}.$$

Since  $\phi_{(i; I)}(X^{2^d-1}) f_i(\bar{z}_t)^{2^d} = \phi_{(i; I)}(X^{2^d-1} \bar{z}_t^{2^d})$  and  $X^{2^d-1} \bar{z}_t^{2^d} \in B_{k-1}(n)$ , we get  $[x] \in [\mathcal{P}_k(n)]$ .

**Case 3.1.2.** If  $d \geq k$ , then  $[x] \in [\mathcal{P}_k(n)]$  for all  $\bar{y} \in (P_{k-1})_m$ .

We have  $\bar{y} = x_i^a f_i(y)$  with  $a = \nu_i(\bar{y})$  and  $y = p_{(i;\emptyset)}(\bar{y}/x_i^a) \in (P_{k-1})_{m-a}$ . The proof proceeds by double induction on  $(i, a)$ . If  $a = 0$ , then by Case 3.1.1,  $[x] \in [\mathcal{P}_k(n)]$  for any  $i$ . Suppose that  $a > 0$  and this case is true for  $a - 1$  and any  $i$ .

If  $i = 1$  and either  $I \neq I_1$  or  $d > k$ , then  $d - r - 1 \geq 1$ . Applying Lemma 3.5(ii) with  $y_0 = x_1^{a-1} f_1(y)$ , we get

$$x \equiv \sum_{2 \leq j \leq k} \phi_{(1; I \cup j)}(X^{2^d-1})(x_1^{a-1} f_1(x_{j-1}y))^{2^d}.$$

From this and the inductive hypothesis, we obtain  $[x] \in [\mathcal{P}_k(n)]$ .

If  $I = I_1$  and  $d = k$ , then  $r = d - 1$ . Using Lemma 3.5(i) with  $y_0 = x_1^{a-1} f_1(y)$  and Lemmas 3.6, 2.14, we get

$$x \equiv \sum_{j=2}^k \phi_{(2; I_2)}(X^{2^k-1})(x_j y_0)^{2^k} = Y_2 y_0^{2^k} \equiv \sum \phi_{(1; J)}(X^{2^k-1})(x_1^a f_1(y))^{2^k},$$

where the last sum runs over all  $J \neq I_1$ . Hence,  $[x] \in [\mathcal{P}_k(n)]$ .

Suppose  $i > 1$  and assume this case has been proved in the subcases  $1, 2, \dots, i-1$ . Then,  $r+1 \leq k-i+1 < k \leq d$ . Applying Lemma 3.5(ii) with  $y_0 = x_i^{a-1} f_i(y)$ , we obtain

$$x \equiv \sum_{1 \leq j < i} \phi_{(j; I \cup i)}(X^{2^d-1})y_j^{2^d} + \sum_{i < j \leq k} \phi_{(i; I \cup j)}(X^{2^d-1})(x_i^{a-1} f_i(x_{j-1}y))^{2^d}.$$

Using the inductive hypothesis, we have  $\phi_{(j; I \cup i)}(X^{2^d-1})y_j^{2^d} \in [\mathcal{P}_k(n)]$  for  $j < i$ , and  $\phi_{(i; I \cup j)}(X^{2^d-1})(x_i^{a-1} f_i(x_{j-1}y))^{2^d} \in [\mathcal{P}_k(n)]$  for  $j > i$ . Hence,  $[x] \in [\mathcal{P}_k(n)]$ .

So, the proposition is proved for  $d \geq k$ . In the remaining part of the proof, we assume that  $d = k - 1$ .

**Case 3.1.3.** If  $I = I_i$ ,  $\bar{y} = f_{i-1}(y)$  with  $y \in (P_{k-1})_m$ ,  $\nu_j(y) = 0$  for  $j \leq i-2$ , and  $2 \leq i \leq 4$ , then  $[x] \in [\mathcal{P}_k(n)]$ .

Since  $y \in (P_{k-1})_m$ , we have  $y \equiv \bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_s$  with  $\bar{z}_t$  monomials in  $\bar{\mathcal{B}}$  and  $\nu_j(\bar{z}_t) = 0$  for  $j \leq i-2$ . Using Lemma 2.14, we get

$$x \equiv \phi_{(i; I_i)}(X^{2^d-1})f_{i-1}(y)^{2^d} \equiv \sum_{1 \leq t \leq s} \phi_{(i; I_i)}(X^{2^d-1})f_{i-1}(\bar{z}_t)^{2^d}.$$

If  $\nu_{i-1}(\bar{z}_t) > 0$ , then  $\phi_{(i; I_i)}(X^{2^d-1})f_{i-1}(\bar{z}_t)^{2^d} = \phi_{(1; I_1)}(X^{2^d-1}\bar{z}_t^{2^d})$ . If  $\nu_{i-1}(\bar{z}_t) = 0$ , then  $f_{i-1}(\bar{z}_t) = f_i(\bar{z}_t)$  and  $\phi_{(i; I_i)}(X^{2^d-1})f_{i-1}(\bar{z}_t)^{2^d} = \phi_{(i; I_i)}(X^{2^d-1}\bar{z}_t^{2^d})$ . Hence,  $[x] \in [\mathcal{P}_k(n)]$ .

**Case 3.1.4.** If  $\bar{y} = x_i^{2^s} f_i(y)$  with  $y \in (P_{k-1})_{m-2^s}$ ,  $\nu_j(y) = 0$  for  $j < i$ ,  $r = \ell(I) < k - i - 1$  and  $i \leq 2$ , then  $[x] \in [\mathcal{P}_k(n)]$ .

According to Lemmas 3.8 and 2.14,  $x_i^{2^s} f_i(y)^{2^d} \equiv \sum x_i f_i(z)$ , where the sum runs over some monomials  $z \in P_{k-1}$  with  $\nu_j(z) = 0$ ,  $j < i$ . So, by using Lemma 2.14, we can assume  $s = 0$ .

Let  $i = 1$ . Since  $r+1 < k-1 = d$ , using Lemma 3.5(ii) with  $y_0 = f_1(y)$ , we have

$$x \equiv \sum_{u=2}^k \phi_{(1; I \cup u)}(X^{2^d-1})(f_1(x_{u-1}y))^{2^d}.$$

Hence, by Case 3.1.1,  $[x] \in [\mathcal{P}_k(n)]$ .

Let  $i = 2$ . Since  $r + 1 < k - 2 < d$ , using Lemma 3.5(ii) with  $y_0 = f_2(y)$ , one gets

$$x \equiv \phi_{(1;I \cup 2)}(X^{2^d-1})(x_1 f_1(y))^{2^d} + \sum_{u=3}^k \phi_{(2;I \cup u)}(X^{2^d-1})(f_2(x_{u-1}y))^{2^d}.$$

Since  $\nu_1(y) = 0$ ,  $f_2(y) = f_1(y)$  and  $\ell(I \cup 2) < k - 2$ , from this equality, Case 3.1.1 and Case 3.1.4 with  $i = 1$ , we obtain  $[x] \in [\mathcal{P}_k(n)]$ .

**Example 3.1.4.** Let  $k = 4, d_1 = 5, d_2 = d_3 = 3$ . Then, we have  $n = 45, m = 3$  and  $\omega = (3, 3, 3, 1, 1)$  is the minimal sequence. Let  $B_3(n)$  be the set of all the admissible monomials of degree  $n$  in  $P_3$ . Then  $\bar{B}_3(m)$  is the set of all the admissible monomials of degree  $m$  in  $P_3$ . Let  $I = \emptyset, y_0 = x_4^2 = f_1(x_3^2) = f_2(x_3^2) \in P_4$ . Denote  $A = \phi_{(1;\emptyset)}(X^7)(x_1 y_0)^8$ ,  $B = \phi_{(2;\emptyset)}(X^7)(x_2 y_0)^8$  and  $z = \phi_{(1;2)}(X^7)(x_1 y_0)^8$ . From the proof of this case, we obtain

$$\begin{aligned} A &\equiv \phi_{(1;2)}(X^7)(x_2 y_0)^8 + \phi_{(1;3)}(X^7)(x_3 y_0)^8 + \phi_{(1;4)}(X^7)(x_4 y_0)^8, \\ B &\equiv z + \phi_{(2;3)}(X^7)(x_3 y_0)^8 + \phi_{(2;4)}(X^7)(x_4 y_0)^8, \\ z &\equiv \phi_{(1;2)}(X^7)(x_2 y_0)^8 + \phi_{(1;(2,3))}(X^7)(x_3 y_0)^8 + \phi_{(1;(2,4))}(X^7)(x_4 y_0)^8. \end{aligned}$$

Since  $x_2 y_0 = x_2 x_4^2$ ,  $x_3 y_0 = x_3 x_4^2$ ,  $x_4 y_0 = x_4^3$  are the admissible monomials, we get  $[A], [B], [x] \in [\mathcal{P}_4(n)]$ . Furthermore,

$$\begin{aligned} A &\equiv x_1 x_2^7 x_3^7 x_4^{30} + x_1 x_2^7 x_3^{14} x_4^{23} + x_1 x_2^{14} x_3^7 x_4^{23}, \\ B &\equiv x_1 x_2^{14} x_3^7 x_4^{23} + x_1^3 x_2^5 x_3^7 x_4^{30} + x_1^3 x_2^5 x_3^{14} x_4^{23} + x_1^7 x_2 x_3^7 x_4^{30} + x_1^7 x_2 x_3^{14} x_4^{23}. \end{aligned}$$

All monomials in the right hand sides of the last equalities are admissible.

**Case 3.1.5.** If  $\bar{y} = x_3^{2^s} f_3(y)$ , with  $y \in (P_{k-1})_{m-2^s}$ ,  $\nu_1(y) = \nu_2(y) = 0$  and  $i = 3$ , then  $[x] \in [\mathcal{P}_k(n)]$ .

According to Lemmas 3.8 and 2.14, we need only to prove this case for  $s = 0$ . Note that, since  $\nu_1(y) = \nu_2(y) = 0$ , we have  $x_3 f_3(y) = f_2(x_2 y)$ . If  $I = I_3$ , then by Case 3.1.3 with  $i = 3$ ,  $[x] \in [\mathcal{P}_k(n)]$ . Suppose  $I \neq I_3$ .

If  $d_{k-2} > d_{k-1}$ , then  $\omega_k(x) = \omega_1(y) + 1 = k - 2$ . Hence,  $\alpha_0(\nu_j(y)) = 1$  for  $j = 3, \dots, k - 1$ . Applying Lemma 3.5(i) with  $y_0 = f_3(y)$  and Theorem 2.12, we get

$$x \equiv \phi_{(1;I)}(X^{2^d-1})(x_1 f_1(y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2 f_2(y))^{2^d}.$$

Hence, by Case 3.1.4,  $[x] \in [\mathcal{P}_k(n)]$ .

Suppose that  $d_{k-2} = d_{k-1}$ . If  $\ell(I) < k - 4$ , then using Lemma 3.5(ii) with  $y_0 = f_3(y) = f_1(y) = f_2(y)$ , one gets

$$\begin{aligned} x &\equiv \phi_{(1;I \cup 3)}(X^{2^d-1})(x_1 f_1(y))^{2^d} + \phi_{(2;I \cup 3)}(X^{2^d-1})(x_2 f_2(y))^{2^d} \\ &\quad + \sum_{v=4}^k \phi_{(3;I \cup v)}(X^{2^d-1})(f_3(x_{v-1}y))^{2^d}. \end{aligned}$$

From this equality and Cases 3.1.1, 3.1.4, we obtain  $[x] \in [\mathcal{P}_k(n)]$ .

If  $\ell(I) = k - 4$ , then  $I = (4, \dots, \hat{u}, \dots, k)$  with  $4 \leq u \leq k$ . Since  $\omega_k(x) = \omega_1(y) + 1 = k - 3$ , we have  $\omega_1(y) = k - 4$ . Hence, there exists uniquely  $3 \leq t < k$  such that  $\alpha_0(\nu_t(y)) = 0$ .

If  $t = u - 1$ , then using Lemma 3.5(i) with  $y_0 = f_3(y)$  and Theorem 2.12, we obtain

$$\begin{aligned} x \equiv & \phi_{(1;I)}(X^{2^d-1})(x_1 f_1(y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2 f_2(y))^{2^d} \\ & + \phi_{(4;I_4)}(X^{2^d-1})(f_3(x_t y))^{2^d}. \end{aligned}$$

By Cases 3.1.3 and 3.1.4, we get  $[x] \in [\mathcal{P}_k(n)]$ .

If  $u = 4 < t + 1$ , then using Lemma 3.5(i) with  $y_0 = f_3(y)$  and Theorem 2.12, we get

$$\begin{aligned} x \equiv & \phi_{(1;I)}(X^{2^d-1})(x_1 f_1(y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2 f_2(y))^{2^d} \\ & + \phi_{(5;I_5)}(X^{2^d-1})(x_5 f_3(x_t y/x_4))^{2^d}. \end{aligned}$$

Applying Lemma 3.5(i) with  $y_0 = f_3(x_t y/x_4)$  and Theorem 2.12, we have

$$\phi_{(5;I_5)}(X^{2^d-1})(x_5 f_3(x_t y/x_4))^{2^d} \equiv \sum_{1 \leq v \leq 3} \phi_{(v;I_5)}(X^{2^d-1})(x_v f_3(x_t y/x_4))^{2^d}.$$

Since  $\ell(I_5) = k - 5 < k - 4$ ,  $\phi_{(3;I_5)}(X^{2^d-1})(x_3 f_3(x_t y/x_4))^{2^d} \in [\mathcal{P}_k(n)]$ . So, combining Case 3.1.4, the above equalities and the fact that  $x_v f_3(x_t y/x_4) = x_v f_v(x_t y/x_4)$  for  $v = 1, 2$ , one gets  $[x] \in [\mathcal{P}_k(n)]$ .

Suppose that  $4 < u \neq t + 1$ . Using Lemma 3.5(i) with  $y_0 = f_3(y)$  and Theorem 2.12, we obtain

$$\begin{aligned} x \equiv & \phi_{(1;I)}(X^{2^d-1})(x_1 f_1(y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2 f_2(y))^{2^d} \\ & + \phi_{(4;I \setminus 4)}(X^{2^d-1})(x_4 f_3(x_t y/x_3))^{2^d}. \end{aligned}$$

Applying Lemma 3.5(i) with  $y_0 = f_3(x_t y/x_3)$  and Theorem 2.12, we have

$$\phi_{(4;I \setminus 4)}(X^{2^d-1})(x_4 f_3(x_t y/x_3))^{2^d} \equiv \sum_{1 \leq v \leq 3} \phi_{(v;I \setminus 4)}(X^{2^d-1})(x_v f_3(x_t y/x_3))^{2^d}.$$

Since  $\ell(I \setminus 4) = k - 5 < k - 4$ ,  $\phi_{(3;I \setminus 4)}(X^{2^d-1})(x_3 f_3(x_t y/x_3))^{2^d} \in [\mathcal{P}_k(n)]$ . So, from the above equalities, Case 3.1.4 and the fact that  $x_v f_3(x_t y/x_3) = x_v f_v(x_t y/x_3)$ , for  $v = 1, 2$ , we get  $[x] \in [\mathcal{P}_k(n)]$ .

**Example 3.1.5.** Let  $k = 4, n, m, B_3(n), A, B, y_0$  be as in Example 3.1.4. Then,

$$C = \phi_{(3;\emptyset)}(X^{15})(x_3 y_0)^8 \equiv A + B + \phi_{(4;\emptyset)}(X^7)(x_4 y_0)^8.$$

Since  $\phi_{(4;\emptyset)}(X^7)(x_4 y_0)^8 = \phi_{(1;I_1)}(X^7 x_3^{24})$  and  $X^7 x_3^{24} = x_1^7 x_2^7 x_3^{31} \in B_3(n)$ , one gets  $[C] \in [\mathcal{P}_4(n)]$  and

$$\begin{aligned} C \equiv & x_1 x_2^7 x_3^7 x_4^{30} + x_1 x_2^7 x_3^{14} x_4^{23} + x_1^3 x_2^5 x_3^7 x_4^{30} + x_1^3 x_2^5 x_3^{14} x_4^{23} \\ & + x_1^7 x_2 x_3^7 x_4^{30} + x_1^7 x_2 x_3^{14} x_4^{23} + x_1^7 x_2^7 x_3^7 x_4^{24}. \end{aligned}$$

**Case 3.1.6.** If  $\nu_1(\bar{y}) = \nu_2(\bar{y}) = 0$  and  $i = 4$ , then  $[x] \in [\mathcal{P}_k(n)]$ .

Since  $\nu_1(\bar{y}) = \nu_2(\bar{y}) = 0$ , we have  $\bar{y} = x_3^b x_4^c f_4(y)$  for suitable  $y \in (P_{k-1})_{m-b-c}$  with  $\nu_j(y) = 0, j \leq 3$ , and  $b = \nu_3(\bar{y}), c = \nu_4(\bar{y})$ . Using Lemmas 3.8 and 2.14, we assume that  $b = 2^s - 1$ .

We prove this case by induction on  $c$ . If  $c = 0$ , then by Case 3.1.1,  $[x] \in [\mathcal{P}_k(n)]$ . Suppose that  $c > 0$  and this case holds for  $c - 1$  and all  $I \subset I_4$ .

If  $I \neq I_4$ , then applying Lemma 3.5(ii) with  $y_0 = x_3^b x_4^{c-1} f_4(y)$ , we have

$$\begin{aligned} x \equiv & \phi_{(1; I \cup 4)}(X^{2^d-1})(x_1 f_1(x_2^b x_3^{c-1} y))^{2^d} + \phi_{(2; I \cup 4)}(X^{2^d-1})(x_2 f_2(x_2^b x_3^{c-1} y))^{2^d} \\ & + \phi_{(3; I \cup 4)}(X^{2^d-1})(x_3^{2^s} f_3(x_3^{c-1} y))^{2^d} + \sum_{u=5}^k \phi_{(4; I \cup u)}(X^{2^d-1})(x_3^b x_4^{c-1} f_4(x_{u-1} y))^{2^d}. \end{aligned}$$

Combining this equality, Cases 3.1.4, 3.1.5 and the inductive hypothesis gives  $[x] \in [\mathcal{P}_k(n)]$ .

If  $I = I_4$ , then applying Lemma 3.5(i) with  $y_0 = x_3^b x_4^{c-1} f_4(y)$  and using Cases 3.1.4, 3.1.5, we obtain

$$\begin{aligned} x \equiv & \phi_{(1; I_4)}(X^{2^d-1})(x_1 f_1(x_2^b x_3^{c-1} y))^{2^d} + \phi_{(2; I_4)}(X^{2^d-1})(x_2 f_2(x_2^b x_3^{c-1} y))^{2^d} \\ & + \phi_{(3; I_4)}(X^{2^d-1})(x_3^{2^s} f_3(x_3^{c-1} y))^{2^d} + Y_5 y_0^{2^d} \equiv Y_5 y_0^{2^d} \pmod{\mathcal{P}_k(n)}. \end{aligned}$$

By Lemmas 3.6 and 2.14,

$$Y_5 y_0^{2^d} \equiv \sum \phi_{(j; J)}(X^{2^d-1})(x_j x_3^b x_4^{c-1} f_4(y))^{2^d},$$

where the sum runs over all  $(j, J)$  with  $1 \leq j < 5, J \subset I_4$  and  $J \neq I_4$ . Since  $J \neq I_4, [\phi_{(4; J)}(X^{2^d-1})(x_3^b x_4^c f_4(y))^{2^d}] \in [\mathcal{P}_k(n)]$ . By Case 3.1.4,

$$[\phi_{(j; J)}(X^{2^d-1})(x_j x_3^b x_4^{c-1} f_4(y))^{2^d}] = [\phi_{(j; J)}(X^{2^d-1})(x_j f_j(x_2^b x_3^{c-1} y))^{2^d}] \in [\mathcal{P}_k(n)],$$

for  $j = 1, 2$ . By Case 3.1.5,

$$[\phi_{(3; J)}(X^{2^d-1})(x_3 x_3^b x_4^{c-1} f_4(y))^{2^d}] = [\phi_{(3; J)}(X^{2^d-1})(x_3^{2^s} f_3(x_3^{c-1} y))^{2^d}] \in [\mathcal{P}_k(n)].$$

Hence,  $[x] \in [\mathcal{P}_k(n)]$ .

**Example 3.1.6.** Let  $k = 4, n, m, B_3(n), C$  be as in Example 3.1.5. Let  $I = \emptyset, y_0 = x_3 x_4, \bar{y} = x_4 y_0 = x_3 x_4^2$ . From the proof of this case, we obtain

$$D = \phi_{(4; \emptyset)}(X^7)(x_4 y_0)^8 \equiv x + y + \phi_{(3; \emptyset)}(X^7)(x_3 y_0)^8,$$

where  $x = \phi_{(1; \emptyset)}(X^7)(x_1 y_0)^8, y = \phi_{(2; \emptyset)}(X^7)(x_2 y_0)^8$ . Since  $x_3 y_0 = x_3^2 x_4 \equiv x_3 x_4^2, \phi_{(3; \emptyset)}(X^7)(x_3 y_0)^8 \equiv C$ . By Case 3.1.4,  $[x], [y] \in [\mathcal{P}_4(n)]$ . Hence,  $[D] = [C] + [x] + [y] \in [\mathcal{P}_4(n)]$ . By a computation analogous to the previous one, we obtain

$$\begin{aligned} D \equiv & x_1 x_2^7 x_3^7 x_4^{30} + x_1 x_2^7 x_3^{15} x_4^{22} + x_1^3 x_2^5 x_3^7 x_4^{30} + x_1^3 x_2^5 x_3^{15} x_4^{22} \\ & + x_1^7 x_2 x_3^7 x_4^{30} + x_1^7 x_2 x_3^{15} x_4^{22} + x_1^7 x_2^7 x_3^7 x_4^{24}. \end{aligned}$$

**Case 3.1.7.** If  $\nu_1(\bar{y}) = \nu_2(\bar{y}) = 0$  and  $i = 3$ , then  $[x] \in [\mathcal{P}(n)]$ .

We have  $\bar{y} = x_3^b f_3(y)$  for suitable  $y \in (P_{k-1})_{m-b}$  with  $\nu_1(y) = \nu_2(y) = 0$ , and  $b = \nu_3(\bar{y})$ . We prove  $[x] \in [\mathcal{P}(n)]$  by induction on  $b$ .

If  $b = 0$ , then by Case 3.1.1,  $[x] \in [\mathcal{P}(n)]$ . Suppose  $b > 0$  and this case holds for  $b - 1$ . If  $\alpha_0(b) = 0$ , then  $\bar{y} = Sq^1(x_3^{b-1} f_3(y)) + x_3^{b-1} f_3(Sq^1(y)) \equiv x_3^{b-1} f_3(Sq^1(y))$ . Hence, using Lemma 2.14 and the inductive hypothesis, one gets  $[x] \in [\mathcal{P}(n)]$ . Now, assume that  $\alpha_0(b) = 1$ .



Since  $x_3^b f_3(y) = f_2(x_2^b y)$ , if  $I = I_3$ , then by Case 3.1.3,  $[x] \in [\mathcal{P}(n)]$ . If  $\ell(I) < k - 4$ , then applying Lemma 3.5(ii) with  $y_0 = x_3^{b-1} f_3(y)$ , we obtain

$$\begin{aligned} x \equiv & \phi_{(1;I \cup 3)}(X^{2^d-1})(x_1 f_1(x_2^{b-1} y))^{2^d} + \phi_{(2;I \cup 3)}(X^{2^d-1})(x_2 f_2(x_2^{b-1} y))^{2^d} \\ & + \sum_{v=4}^k \phi_{(3;I \cup v)}(X^{2^d-1})(x_3^{b-1} f_3(x_{v-1} y))^{2^d}. \end{aligned}$$

Using Case 3.1.4 and the inductive hypothesis, one gets  $[x] \in [\mathcal{P}(n)]$ .

Suppose that  $\ell(I) = k - 4$ . Then  $I = (4, \dots, \hat{u}, \dots, k)$ , with  $4 \leq u \leq k$ .

If  $d_{k-2} > d_{k-1}$ , then  $\omega_k(x) = \omega_1(y) + \alpha_0(b) = k - 2$ . Hence,  $\alpha_0(\nu_j(y)) = 1$  for  $j = 3, \dots, k - 1$ . Applying Lemma 3.5(i) with  $y_0 = x_3^{b-1} f_3(y)$  and Theorem 2.12, we get

$$x \equiv \phi_{(1;I)}(X^{2^d-1})(x_1 f_1(x_2^{b-1} y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2 f_2(x_2^{b-1} y))^{2^d}.$$

Hence, by Case 3.1.4, we obtain  $[x] \in [\mathcal{P}(n)]$ .

Suppose  $d_{k-2} = d_{k-1}$ . Since  $\omega_k(x) = \omega_1(y) + \alpha_0(b) = k - 3$ , we have  $\omega_1(y) = k - 4$ . Hence, there exists uniquely  $3 \leq t \leq k - 1$  such that  $\alpha_0(\nu_t(y)) = 0$ .

If  $t = u - 1$ , then using Lemma 3.5(i) with  $y_0 = x_3^{b-1} f_3(y)$  and Theorem 2.12, we have

$$\begin{aligned} x \equiv & \phi_{(1;I)}(X^{2^d-1})(x_1 f_1(x_2^{b-1} y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2 f_2(x_2^{b-1} y))^{2^d} \\ & + \phi_{(4;I_4)}(X^{2^d-1})(x_3^{b-1} f_3(x_t y))^{2^d}. \end{aligned}$$

From this equality, Case 3.1.4 and 3.1.6, we get  $[x] \in [\mathcal{P}(n)]$ .

If  $u = 4 < t + 1$ , then using Lemma 3.5(i) with  $y_0 = x_3^{b-1} f_3(y)$  and Theorem 2.12, we obtain

$$\begin{aligned} x \equiv & \phi_{(1;I)}(X^{2^d-1})(x_1 f_1(x_2^{b-1} y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2 f_2(x_2^{b-1} y))^{2^d} \\ & + \phi_{(5;I_5)}(X^{2^d-1})(x_3^{b-1} f_3(x_t y))^{2^d}. \end{aligned}$$

Applying Lemma 3.5(i) with  $y_0 = x_3^{b-1} f_3(x_t y/x_4)$  and Theorem 2.12, we have

$$\phi_{(5;I_5)}(X^{2^d-1})(x_3^{b-1} f_3(x_t y))^{2^d} \equiv \sum_{1 \leq v \leq 3} \phi_{(v;I_5)}(X^{2^d-1})(x_v x_3^{b-1} f_3(x_t y/x_4))^{2^d}.$$

Since  $\ell(I_5) = k - 5 < k - 4$ ,  $\phi_{(3;I_5)}(X^{2^d-1})(x_3^b f_3(x_t y/x_4))^{2^d} \in [\mathcal{P}(n)]$ . Hence, combining the above equalities, Case 3.1.4 and the fact that  $x_v x_3^{b-1} f_3(x_t y/x_4) = x_v f_v(x_2^{b-1} x_t y/x_4)$ , for  $v = 1, 2$ , one gets  $[x] \in [\mathcal{P}(n)]$ .

Suppose that  $4 < u \neq t + 1$ . Using Lemma 3.5(i) with  $y_0 = x_3^{b-1} f_3(y)$  and Theorem 2.12, we obtain

$$\begin{aligned} x \equiv & \phi_{(1;I)}(X^{2^d-1})(x_1 f_1(x_2^{b-1} y))^{2^d} + \phi_{(2;I)}(X^{2^d-1})(x_2 f_2(x_2^{b-1} y))^{2^d} \\ & + \phi_{(4;I \setminus 4)}(X^{2^d-1})(x_3^{b-1} f_3(x_t y))^{2^d}. \end{aligned}$$

From the above equalities, Cases 3.1.4 and 3.1.6, we get  $[x] \in [\mathcal{P}(n)]$ .

**Example 3.1.7.** Let  $k = 4, n, m, B_3(n), D$  be as in Example 3.1.6. Let  $I = \emptyset, y_0 = x_3^2 = x_3^2 f_3(x_3^0) \in P_4$ . This case is entry with  $b = 3, u = 4, t = 3 = u - 1$ . Then,  $E = \phi_{(3;\emptyset)}(X^7)(x_3 y_0)^8 \equiv x + y + \phi_{(4;\emptyset)}(X^7)(x_4 y_0)^8$ , where  $x = \phi_{(1;\emptyset)}(X^7)(x_1 y_0)^8$ ,  $y = \phi_{(2;\emptyset)}(X^7)(x_2 y_0)^8$ . Since  $x_4 y_0 = x_3^2 x_4 \equiv x_3 x_4^2$ , we have  $\phi_{(4;\emptyset)}(X^7)(x_4 y_0)^8 \equiv D$ .

By Case 3.1.4,  $[x], [y] \in [\mathcal{P}_4(n)]$ . Hence,  $[E] = [D] + [x] + [y] \in [\mathcal{P}_4(n)]$ . By a computation analogous to the previous one, we obtain

$$\begin{aligned} E \equiv & x_1 x_2^7 x_3^7 x_4^{30} + x_1 x_2^7 x_3^{30} x_4^7 + x_1^3 x_2^5 x_3^7 x_4^{30} + x_1^3 x_2^5 x_3^{30} x_4^7 \\ & + x_1^7 x_2 x_3^7 x_4^{30} + x_1^7 x_2 x_3^{30} x_4^7 + x_1^7 x_2^7 x_3^7 x_4^{24}. \end{aligned}$$

Here, the monomials in the right hand sides of the last relation are admissible.

**Case 3.1.8.** If  $\bar{y} = x_2^{2^s} f_2(y)$  for  $y \in (P_{k-1})_{m-2^s}$  with  $\nu_1(y) = 0$  and  $i = 2$ , then  $[x] \in [\mathcal{P}(n)]$ .

It suffices to prove this case for  $s = 0$ . If  $\ell(I) < k - 3$ , then by Case 3.1.4,  $[x] \in [\mathcal{P}(n)]$ . Since  $x_2 f_2(y) = f_1(x_1 y)$ , if  $I = I_2$ , then by Case 3.1.3,  $[x] \in [\mathcal{P}(n)]$ .

Suppose  $\ell(I) = k - 3$ . Then,  $I = (3, \dots, \hat{u}, \dots, k)$  with  $3 \leq u \leq k$ .

If  $u = 3$ , then using Lemma 3.5(i) with  $y_0 = f_2(y)$ , we get

$$\begin{aligned} x \equiv & \phi_{(1;I_3)}(X^{2^d-1})(x_1 f_1(y))^{2^d} + \phi_{(3;I_3)}(X^{2^d-1})(f_2(x_2 y))^{2^d} \\ & + \sum_{v=4}^k \phi_{(4;I_4)}(X^{2^d-1})(f_2(x_{v-1} y))^{2^d}. \end{aligned}$$

If  $u > 3$ , then using Lemma 3.5(i) with  $y_0 = f_2(y)$ , we get

$$x \equiv \phi_{(1;I)}(X^{2^d-1})(x_1 f_1(y))^{2^d} + \sum_{3 \leq v \leq k} \phi_{(3;I \cup v \setminus 3)}(X^{2^d-1})(f_2(x_{v-1} y))^{2^d}.$$

Since  $\nu_1(f_2(x_{v-1} y)) = \nu_2(f_2(x_{v-1} y)) = 0$ , for  $3 \leq v \leq k$ , combining the above equalities, Cases 3.1.3, 3.1.4, 3.1.6 and 3.1.7, we obtain  $[x] \in [\mathcal{P}(n)]$ .

**Example 3.1.8.** Let  $k = 4, n, m, B_3(n), E$  be as in Example 3.1.6. Let  $I = (3), y_0 = x_2^2 = x_2^2 f_3(x_3^0) \in P_4$ . This case is entry with  $u = 4, t = 3 = u - 1$ . Then,  $F = \phi_{(2;3)}(X^7)(x_2 y_0)^8 \equiv x + E + y$ , where  $x = \phi_{(1;3)}(X^7)(x_1 y_0)^8$ ,  $y = \phi_{(3;4)}(X^7)(x_4 y_0)^8$ . Since  $x_4 y_0 = x_3^2 x_4 \equiv x_3 x_4^2$ , we have  $y \equiv \phi_{(1;I_1)}(X^7(x_2 x_3^2)^8) = x_1^7 x_2^7 x_3^9 x_4^{22}$ . By Case 3.1.4,  $[x] \in [\mathcal{P}_4(n)]$ . Hence,  $[F] = [x] + [E] + [y] \in [\mathcal{P}_4(n)]$ . By a computation analogous to the previous one, we obtain

$$\begin{aligned} F \equiv & x_1 x_2^7 x_3^7 x_4^{30} + x_1^3 x_2^5 x_3^7 x_4^{30} + x_1^3 x_2^5 x_3^{30} x_4^7 + x_1^3 x_2^7 x_3^{13} x_4^{22} + x_1^3 x_2^{13} x_3^{22} x_4^7 \\ & + x_1^7 x_2 x_3^7 x_4^{30} + x_1^7 x_2 x_3^{30} x_4^7 + x_1^7 x_2^7 x_3^7 x_4^{24} + x_1^7 x_2^7 x_3^9 x_4^{22}. \end{aligned}$$

Here, the monomials in the right hand sides of the last relation are admissible.

**Case 3.1.9.** If  $\nu_1(\bar{y}) = 0$  and  $i = 3$ , then  $[x] \in [\mathcal{P}(n)]$ .

We have  $\bar{y} = x_2^a x_3^b f_3(y)$  for suitable  $y \in (P_{k-1})_{m-a-b}$  with  $\nu_1(y) = \nu_2(y) = 0$ ,  $a = \nu_2(\bar{y}), b = \nu_3(\bar{y})$ . Using Lemmas 3.8 and 2.14, we can assume that  $a = 2^s - 1$ . We prove this case by induction on  $b$ . If  $b = 0$ , then by Case 3.1.1, this case is true. Suppose that  $b > 0$  and this case is true for  $b - 1$ .

If  $I \neq I_3$ , then using Lemma 3.5(ii) with  $y_0 = x_2^a x_3^{b-1} f_3(y)$ , we get

$$\begin{aligned} x \equiv & \phi_{(1;I \cup 3)}(X^{2^d-1})(x_1 f_1(x_1^a x_2^{b-1} y))^{2^d} + \phi_{(2;I \cup 3)}(X^{2^d-1})(x_2^{2^s} f_2(x_2^{b-1} y))^{2^d} \\ & + \sum_{v=4}^k \phi_{(3;I \cup v)}(X^{2^d-1})(x_2^a x_3^{b-1} f_3(x_{v-1} y))^{2^d}. \end{aligned}$$

From this equality, Case 3.1.4 and the inductive hypothesis we obtain  $[x] \in [\mathcal{P}(n)]$ .

If  $I = I_3$ , then using Lemma 3.5(i) with  $y_0 = x_2^a x_3^{b-1} f_3(y)$ , and Case 3.1.4, we have

$$\begin{aligned} x &\equiv \phi_{(1;I_3)}(X^{2^d-1})(x_1 f_1(x_1^a x_2^{b-1} y))^{2^d} \\ &\quad + \phi_{(2;I_3)}(X^{2^d-1})(x_2^{2^s} f_2(x_2^{b-1} y))^{2^d} + Y_4 y_0^{2^d} \equiv Y_4 y_0^{2^d} \pmod{\mathcal{P}(n)}. \end{aligned}$$

Using Lemmas 3.6 and 2.14, we get

$$Y_4 y_0^{2^d} \equiv \sum_{(j;J)} \phi_{(j;J)}(X^{2^d-1})(x_j x_2^a x_3^{b-1} f_3(y))^{2^d},$$

where the last sum runs over some  $(j;J)$  with  $1 \leq j < 4$ ,  $J \subset I_3$  and  $J \neq I_3$ . Since  $J \neq I_3$ ,  $\phi_{(3;J)}(X^{2^d-1})(x_2^a x_3^b f_3(y))^{2^d} \in [\mathcal{P}(n)]$ . By Cases 3.1.4 and 3.1.8,

$$\begin{aligned} \phi_{(1;J)}(X^{2^d-1})(x_1 x_2^a x_3^{b-1} f_3(y))^{2^d} &= \phi_{(1;J)}(X^{2^d-1})(x_1 f_1(x_1^a x_2^{b-1} y))^{2^d} \in [\mathcal{P}(n)], \\ \phi_{(2;J)}(X^{2^d-1})(x_2 x_2^a x_3^{b-1} f_3(y))^{2^d} &= \phi_{(2;J)}(X^{2^d-1})(x_2^{2^s} f_2(x_2^{b-1} y))^{2^d} \in [\mathcal{P}(n)]. \end{aligned}$$

This case is proved.

**Example 3.1.9.** Let  $k = 4, n, m, B_3(n), F$  be as in Example 3.1.8. Let  $I = I_3 = (4), y_0 = x_2 x_4, \bar{y} = x_3 y_0$ . Since  $I = I_3$ ,  $G = \phi_{(3;4)}(X^7)(x_3 y_0)^8 \equiv Y + Z + Y_4 y_0^8$ , where  $Y = \phi_{(1;4)}(X^7)(x_1 y_0)^8$ ,  $Z = \phi_{(2;4)}(X^7)(x_2^2 x_4)^8$ . By Case 3.1.4,  $[Y], [Z] \in [\mathcal{P}_4(n)]$ . Since  $Y_4 y_0^8 = \phi_{(4;\emptyset)}(X^7)(x_2 x_4)^8$ , we have  $Y_4 y_0^8 \equiv y + z + w$ , where

$$y = \phi_{(1;\emptyset)}(X^7)(x_1 x_2 x_4)^8, \quad z = \phi_{(2;\emptyset)}(X^7)(x_2^2 x_4)^8, \quad w = \phi_{(3;\emptyset)}(X^7)(x_2 x_3 x_4)^8.$$

By Case 3.1.4,  $[y], [z] \in [\mathcal{P}_4(n)]$ . We have

$$w \equiv \phi_{(1;3)}(X^7)(x_1 x_2 x_4)^8 + \phi_{(2;3)}(X^7)(x_2^2 x_4)^8 + \phi_{(3;4)}(X^7)(x_2 x_4^2)^8.$$

Using Cases 3.1.1 and 3.1.4, one gets  $[w] \in [\mathcal{P}_4(n)]$ . Hence,  $[x] = [Y] + [Z] + [y] + [z] + [w] \in [\mathcal{P}_4(n)]$ . By a computation analogous to the previous one, we obtain

$$\begin{aligned} G &= x_1^7 x_2^{15} x_3^9 x_4^{14} \equiv x_1^3 x_2^{15} x_3^{13} x_4^{14} + x_1^7 x_2^7 x_3^9 x_4^{22} + x_1^3 x_2^7 x_3^{13} x_4^{22} \\ &\quad + x_1^3 x_2^{15} x_3^5 x_4^{22} + x_1^7 x_2^{15} x_3 x_4^{22} + x_1^7 x_2^7 x_3 x_4^{30} + x_1^3 x_2^7 x_3^5 x_4^{30}. \end{aligned}$$

All monomials in the right hand sides of the last relation are admissible.

Now, let  $I = \emptyset, y_1 = x_2 x_3, \bar{y} = x_3 y_1$ . Then,  $H = \phi_{(3;\emptyset)}(X^7)(x_3 y_1)^8 \equiv a + F + G$ , where  $a = \phi_{(1;3)}(X^7)(x_1 y_1)^8$ . By Case 3.1.4,  $[a] \in [\mathcal{P}_4(n)]$ . Hence,  $[H] = [a] + [F] + [G] \in [\mathcal{P}_4(n)]$ . By a computation analogous to the previous one, we obtain

$$\begin{aligned} H &= x_1 x_2^7 x_3^7 x_4^{30} + x_1 x_2^{15} x_3^{22} x_4^7 + x_1^3 x_2^5 x_3^7 x_4^{30} + x_1^3 x_2^5 x_3^{30} x_4^7 \\ &\quad + x_1^3 x_2^7 x_3^5 x_4^{30} + x_1^3 x_2^{15} x_3^5 x_4^{22} + x_1^7 x_2 x_3^7 x_4^{30} + x_1^7 x_2 x_3^{30} x_4^7 \\ &\quad + x_1^7 x_2^7 x_3 x_4^{30} + x_1^7 x_2^7 x_3^7 x_4^{24} + x_1^7 x_2^{15} x_3 x_4^{22}. \end{aligned}$$

All monomials in the right hand sides of the last relation are admissible.

**Case 3.1.10.** If  $\nu_1(\bar{y}) = 0$  and  $i = 2$ , then  $[x] \in [\mathcal{P}(n)]$ .

We have  $\bar{y} = x_2^a f_2(y)$  for suitable  $y \in (P_{k-1})_{m-a}$  with  $\nu_1(y) = 0$  and  $a = \nu_1(\bar{y})$ . We prove  $[x] \in [\mathcal{P}(n)]$  by induction on  $a$ . If  $a = 0$ , then by Case 3.1.1,  $[x] \in [\mathcal{P}(n)]$ . Suppose that  $a > 0$  and this case holds for  $a - 1$ .

Since  $x_2^a f_2(y) = f_1(x_1^a y)$ , if  $I = I_2$ , then by Case 3.1.3,  $[x] \in [\mathcal{P}(n)]$ . If  $\ell(I) < k - 3$ , then applying Lemma 3.5(ii) with  $y_0 = x_2^{a-1} f_2(y)$ , we have

$$x \equiv \phi_{(1; I \cup 2)}(X^{2^d-1})(x_1 f_1(x_1^{a-1} y))^{2^d} + \sum_{v=3}^k \phi_{(2; I \cup v)}(X^{2^d-1})(x_2^{a-1} f_2(x_{v-1} y))^{2^d}.$$

Using Case 3.1.4 and the inductive hypothesis, we get  $[x] \in [\mathcal{P}(n)]$ .

Suppose that  $\ell(I) = k - 3$ . Then  $I = (3, \dots, \hat{u}, \dots, k)$  with  $3 \leq u \leq k$ .

If  $u = 3$ , then  $I = I_3$ . Applying Lemma 3.5(i) with  $y_0 = x_2^{a-1} f_2(y)$ , we obtain

$$\begin{aligned} x \equiv \phi_{(1; I_3)}(X^{2^d-1})(x_1 f_1(x_1^{a-1} y))^{2^d} &+ \phi_{(3; I_3)}(X^{2^d-1})(x_2^{a-1} f_2(x_2 y))^{2^d} \\ &+ \sum_{v=4}^k \phi_{(4; I_4)}(X^{2^d-1})(x_v x_2^{a-1} f_2(y))^{2^d}. \end{aligned}$$

Applying Lemma 3.6 and Lemma 2.14, one gets

$$\begin{aligned} \sum_{v=4}^k \phi_{(4; I_4)}(X^{2^d-1})(x_v x_2^{a-1} f_2(y))^{2^d} &= Y_4 y_0^{2^d} \\ &\equiv \sum_{(j; J)} \phi_{(j; J)}(X^{2^d-1})(x_j x_2^{a-1} f_2(y))^{2^d}, \end{aligned}$$

where the last sum runs over some  $(j; J)$  with  $1 \leq j < 4$ ,  $J \subset I_3$  and  $J \neq I_3$ .

Since  $\ell(J) < \ell(I_3) = k - 3$ , from the above equalities and Cases 3.1.4, 3.1.9, we have  $[x] \in [\mathcal{P}(n)]$ .

If  $u > 3$ , applying Lemma 3.5(i) with  $y_0 = x_2^{a-1} f_2(y)$ , we get

$$x \equiv \phi_{(1; I)}(X^{2^d-1})(x_1 f_1(x_1^{a-1} y))^{2^d} + \sum_{v=3}^k \phi_{(3; I \cup v)}(X^{2^d-1})(x_2^{a-1} f_2(x_{v-1} y))^{2^d}.$$

From the last equality, Cases 3.1.4 and 3.1.9, we have  $[x] \in [\mathcal{P}(n)]$ .

**Example 3.1.10.** Let  $k = 4, n, m, B_3(n), H$  be as in Example 3.1.9. Let  $I = I_3 = (3), y_0 = x_2^2, \bar{y} = x_2 y_0 = x_2^3$ . Then,  $K := \phi_{(2; 3)}(X^7)(x_2 y_0)^8 \equiv a + H + b$ , where  $a = \phi_{(1; 3)}(X^7)(x_1 y_0)^8$ ,  $b = \phi_{(3; 4)}(X^7)(x_4 y_0)^8$ . By Cases 3.1.1 and 3.1.4,  $[a], [b] \in [\mathcal{P}_4(n)]$ . Hence,  $[K] = [a] + [H] + [b] \in [\mathcal{P}_4(n)]$ . By a simple computation, we have

$$\begin{aligned} K &= x_1 x_2^7 x_3^7 x_4^{30} + x_1^3 x_2^5 x_3^7 x_4^{30} + x_1^3 x_2^5 x_3^{30} x_4^7 + x_1^3 x_2^7 x_3^5 x_4^{30} + x_1^3 x_2^{29} x_3^6 x_4^7 \\ &+ x_1^7 x_2 x_3^7 x_4^{30} + x_1^7 x_2 x_3^{30} x_4^7 + x_1^7 x_2^7 x_3 x_4^{30} + x_1^7 x_2^7 x_3^7 x_4^{24}. \end{aligned}$$

All monomials in the right hand sides of the last relation are admissible.

**Case 3.1.11.** If  $\bar{y} = x_1^{2^s} f_1(y)$  with  $y \in (P_{k-1})_{m-2^s}$  and  $i = 1$ , then  $[x] \in [\mathcal{P}(n)]$ .

By Lemmas 3.8 and 2.14, we need only to prove this case for  $s = 0$ . Note that  $r = \ell(I) < d = k - 1$ . If  $r < k - 2$ , then by Case 3.1.4,  $[x] \in [\mathcal{P}(n)]$ . If  $r = k - 2$ , then  $I = (2, \dots, \hat{u}, \dots, k)$  with  $2 \leq u \leq k$ .

If  $u = 2$ , then  $I = I_2$ . Applying Lemma 3.5(i) with  $y_0 = f_1(y)$ , one gets

$$x \equiv \phi_{(2; I_2)}(X^{2^d-1})(f_1(x_1 y))^{2^d} + \sum_{v=3}^k \phi_{(3; I_3)}(X^{2^d-1})(f_1(x_{v-1} y))^{2^d}.$$

By Case 3.1.3,  $\phi_{(2;I_2)}(X^{2^d-1})(f_1(x_1y))^{2^d} \in [\mathcal{P}(n)]$ . Since  $\nu_1(f_1(x_1y))^{2^d} = 0$ , by Case 3.1.9,  $\phi_{(3;I_3)}(X^{2^d-1})(f_1(x_{v-1}y))^{2^d} \in [\mathcal{P}(n)]$ . Hence,  $x \in [\mathcal{P}(n)]$ .

If  $u > 2$ , then applying Lemma 3.5(i) with  $y_0 = f_1(y)$ , we obtain

$$x \equiv \sum_{2 \leq v \leq k} \phi_{(2;I \setminus 2)}(X^{2^d-1})(f_1(x_{v-1}y))^{2^d}.$$

Since  $\nu_1(f_1(x_{v-1}y)) = 0$ , this equality and Case 3.1.10 imply  $[x] \in [\mathcal{P}(n)]$ .

**Example 3.1.11.** Let  $k = 4, n, m, B_3(n), K$  be as in Example 3.1.10 and let  $H$  be as in Example 3.1.8. Let  $I = I_3 = (2, 3), y_0 = x_2^2, \bar{y} = x_1y_0 = x_1x_2^2$ . Then,  $L := \phi_{(1;(2,3))}(X^7)(x_1y_0)^8 \equiv K + F + a$ , where  $a = \phi_{(2;I_2)}(X^7)(x_4y_0)^8 \equiv \phi_{(1;I_1)}(X^7)(x_1x_3^2)^8$ . Hence,  $[L] = [K] + [F] + [a] \in [\mathcal{P}_4(n)]$ . By a simple computation, we have

$$\begin{aligned} L = & x_1^3x_2^7x_3^5x_4^{30} + x_1^3x_2^7x_3^{13}x_4^{22} + x_1^3x_2^{13}x_3^{22}x_4^7 + x_1^3x_2^{29}x_3^6x_4^7 \\ & + x_1^7x_2^7x_3x_4^{30} + x_1^7x_2^7x_3^9x_4^{22} + x_1^7x_2^{11}x_3^5x_4^{22}. \end{aligned}$$

All monomials in the right hand sides of the last relation are admissible.

**Case 3.1.12.** If  $i = 2$ , then  $[x] \in [\mathcal{P}(n)]$ .

We have  $\bar{y} = x_1^a x_2^b f_2(y)$  for suitable  $y \in (P_{k-1})_{m-a-b}$  with  $\nu_1(y) = 0$ , and  $a = \nu_1(\bar{y}), b = \nu_2(\bar{y})$ . We prove this case by induction on  $b$ . By using Lemmas 3.8 and 2.14, we can assume that  $a = 2^s - 1$ .

If  $b = 0$ , then by Case 3.1.1,  $[x] \in [\mathcal{P}(n)]$ . Suppose that  $b > 0$  and this case is true for  $b - 1$ .

If  $I \neq I_2$ , then applying Lemma 3.5(ii) with  $y_0 = x_1^a x_2^{b-1} f_2(y)$ , we get

$$\begin{aligned} x \equiv & \phi_{(1;I \cup 2)}(X^{2^d-1})(x_1^{2^s} f_1(x_1^{b-1}y))^{2^d} \\ & + \sum_{3 \leq v \leq k} \phi_{(2;I \cup v)}(X^{2^d-1})(x_1^a x_2^{b-1} f_2(x_{v-1}y))^{2^d}. \end{aligned}$$

This equality, Case 3.1.11 and the inductive hypothesis imply  $[x] \in [\mathcal{P}(n)]$ .

If  $I = I_2$ , then applying Lemma 3.5(i) with  $y_0 = x_1^a x_2^{b-1} f_2(y)$  and using Case 3.1.4, we get

$$x \equiv \phi_{(1;I_2)}(X^{2^d-1})(x_1^{2^s} f_1(x_1^{b-1}y))^{2^d} + Y_3 y_0^{2^d} \equiv Y_3 y_0^{2^d} \pmod{[\mathcal{P}(n)]}.$$

By Lemmas 3.6 and 2.14, we have

$$Y_3 y_0^{2^d} \equiv \sum_{(j;J)} \phi_{(j;J)}(X^{2^d-1})(x_j x_1^a x_2^{b-1} f_2(y))^{2^d},$$

where the last sum runs over some  $(j; J)$  with  $j = 1, 2, J \subset I_2$  and  $J \neq I_2$ . Since  $J \neq I_2$ ,  $\phi_{(2;J)}(X^{2^d-1})(x_1^a x_2^b f_2(y))^{2^d} \in [\mathcal{P}(n)]$ . By Case 3.1.11,

$$\phi_{(1;J)}(X^{2^d-1})(x_1 x_1^a x_2^{b-1} f_2(y))^{2^d} = \phi_{(1;J)}(X^{2^d-1})(x_1^{2^s} f_1(x_1^{b-1}y))^{2^d} \in [\mathcal{P}(n)].$$

This case is proved.

**Example 3.1.12.** Let  $k = 4, n, m, B_3(n)$  be as in Example 3.1.4 and let  $H$  be as in Example 3.1.8. Let  $I = I_3 = \emptyset, y_0 = x_1x_3, \bar{y} = x_2y_0 = x_1x_2x_3$ . Then,  $M := \phi_{(2;\emptyset)}(X^7)(x_1y_0)^8 \equiv a + b + c$ , where  $a = \phi_{(1;2)}(X^7)(x_1y_0)^8, b = \phi_{(2;3)}(X^7)(x_3y_0)^8,$

$c = \phi_{(2;4)}(X^7)(x_4 y_0)^8$ . By Cases 3.1.1 and 3.1.4,  $[a], [b], [c] \in [\mathcal{P}_4(n)]$ . Hence,  $[M] \in [\mathcal{P}_4(n)]$ . By a simple computation, we have

$$M = x_1 x_2^{14} x_3^{23} x_4^7 + x_1^3 x_2^5 x_3^{15} x_4^{22} + x_1^3 x_2^5 x_3^{30} x_4^7 + x_1^{15} x_2 x_3^{22} x_4^7 + x_1^{15} x_2 x_3^{15} x_4^{14}.$$

All monomials in the right hand sides of the last relation are admissible.

**Case 3.1.13.** If  $i = 1$ , then  $[x] \in [\mathcal{P}(n)]$ .

We have  $\bar{y} = x_1^a f_1(y)$  for suitable  $y \in (P_{k-1})_{m-a}$  and  $a = \nu_1(\bar{y})$ . We prove this case by induction on  $a$ .

If  $a = 0$ , then by Case 3.1.1,  $[x] \in [\mathcal{P}(n)]$ . Suppose that  $a > 0$  and this case holds for  $a - 1$ .

Note that  $r = \ell(I) \leq d - 1 = k - 2$ . If  $r < k - 2$ , then applying Lemma 3.5(ii) with  $y_0 = x_1^{a-1} f_1(y)$ , we get

$$x \equiv \sum_{v=2}^k \phi_{(1; I \cup v)}(X^{2^d-1})(x_1^{a-1} f_1(x_{v-1} y))^2.$$

Hence, by the inductive hypothesis, we obtain  $[x] \in [\mathcal{P}(n)]$ .

Suppose that  $r = k - 2$ . Then,  $I = (2, \dots, \hat{u}, \dots, k)$  with  $2 \leq u \leq k$ . If  $u = 2$ , then applying Lemma 3.5(i) with  $y_0 = x_1^{a-1} f_1(y)$ , Lemma 2.14 and Case 3.1.12, we get

$$x \equiv \phi_{(2; I_2)}(X^{2^d-1})(x_1^{a-1} f_1(x_1 y))^2 + Y_3 y_0^{2^d} \equiv Y_3 y_0^{2^d} \pmod{\mathcal{P}_k(n)}.$$

By Lemmas 3.6 and 2.14, we have

$$Y_3 y_0^{2^d} \equiv \sum_{(j; J)} \phi_{(j; J)}(X^{2^d-1})(x_j x_1^{a-1} f_1(y))^2,$$

where the last sum runs over some  $(j; J)$  with  $j = 1, 2$ ,  $J \subset I_2$  and  $J \neq I_2$ . By Case 3.1.12,  $\phi_{(2; J)}(X^{2^d-1})(x_2 x_1^{a-1} f_1(y))^2 \in [\mathcal{P}(n)]$ . Since  $J \neq I_2$ , we have  $\phi_{(1; J)}(X^{2^d-1})(x_1^a f_1(y))^2 \in [\mathcal{P}(n)]$ . Hence,  $x \in [\mathcal{P}(n)]$ .

If  $u > 2$ , then applying Lemma 3.5(i) with  $y_0 = x_1^{a-1} f_1(y)$ , we get

$$x \equiv \sum_{2 \leq v \leq k} \phi_{(2; I \cup v \setminus 2)}(X^{2^d-1})(x_1^{a-1} f_1(x_{v-1} y))^2.$$

From the last equality and Case 3.1.12, we obtain  $[x] \in [\mathcal{P}(n)]$ .

**Example 3.1.13.** Let  $k = 4, n, m, B_3(n), L$  be as in Example 3.1.11. Let  $I = I_3 = (2, 3)$ ,  $y_0 = x_1^2, \bar{y} = x_1 y_0 = x_1^3$ . Then,  $N := \phi_{(1; (2, 3))}(X^7)(x_1 y_0)^8 \equiv L + a + b$ , where  $a = \phi_{(2; 3)}(X^7)(x_3 y_0)^8 \equiv \phi_{(1; I_1)}(X^7(x_1 x_3^2)^8)$  and  $b = \phi_{(2; I_2)}(X^7)(x_4 y_0)^8 \equiv \phi_{(1; I_1)}(X^7(x_1 x_4^2)^8)$ . Hence,  $[N] = [L] + [a] + [b] \in [\mathcal{P}_4(n)]$ . By a simple computation, we have

$$\begin{aligned} N = & x_1^3 x_2^7 x_3^5 x_4^{30} + x_1^3 x_2^7 x_3^{13} x_4^{22} + x_1^3 x_2^{13} x_3^{22} x_4^7 + x_1^3 x_2^{29} x_3^6 x_4^7 \\ & + x_1^7 x_2^7 x_3 x_4^{30} + x_1^7 x_2^7 x_3^9 x_4^{22} + x_1^7 x_2^{11} x_3^5 x_4^{22} + x_1^{15} x_2 x_3^{22} x_4^7 + x_1^{15} x_2^3 x_3^5 x_4^{22}. \end{aligned}$$

All monomials in the right hand sides of the last relation are admissible.

**Case 3.1.14.**  $[x] \in [\mathcal{P}(n)]$  for all  $\bar{y} \in (P_{k-1})_m$ .

We have  $\bar{y} = x_i^a f_i(y)$  with  $a = \nu_i(\bar{y})$  and  $y = p_{(i; \emptyset)}(\bar{y}/x_i^a) \in (P_{k-1})_{m-a}$ . We prove this case by double induction on  $(i, a)$ . If  $a = 0$ , then by Case 3.1.1,  $[x] \in [\mathcal{P}(n)]$  for all  $i$ . Suppose  $a > 0$  and this case holds for  $a - 1$  and all  $i$ .

If  $i = 1, 2$ , then by Cases 3.1.12 and 3.1.13,  $[x] \in [\mathcal{P}(n)]$  for all  $a$ . Suppose  $i > 2$  and assume this case already proved in the subcases  $1, 2, \dots, i-1$ . Then,  $r+1 \leq k-i+1 < k-1 = d$ . Applying Lemma 3.5(ii) with  $y_0 = x_i^{a-1}f_i(y)$ , we obtain

$$x \equiv \sum_{1 \leq j < i} \phi_{(j; I \cup i)}(X^{2^d-1})y_j^{2^d} + \sum_{i < j \leq k} \phi_{(i; I \cup j)}(X^{2^d-1})(x_i^{a-1}f_i(x_{j-1}y))^{2^d}.$$

Using the inductive hypothesis, we have  $\phi_{(j; I \cup i)}(X^{2^d-1})y_j^{2^d} \in [\mathcal{P}(n)]$  for  $j < i$ , and  $\phi_{(i; I \cup j)}(X^{2^d-1})(x_i^{a-1}f_i(x_{j-1}y))^{2^d} \in [\mathcal{P}(n)]$  for  $j > i$ . Hence,  $[x] \in [\mathcal{P}(n)]$ .

Now we prove that the set  $[B_k(n)]$  is linearly independent in  $QP_k$ . Suppose there is a linear relation

$$\mathcal{S} = \sum_{((i; I), z) \in \mathcal{N}_k \times B_{k-1}(n)} \gamma_{(i; I), z} \phi_{(i; I)}(z) \equiv 0, \quad (3.4)$$

where  $\gamma_{(i; I), z} \in \mathbb{F}_2$ .

If  $d \geq k$ , then by induction on  $\ell(I)$ , we can show that  $\gamma_{(i; I), z} = 0$ , for all  $(i; I) \in \mathcal{N}_k$  and  $z \in B_{k-1}(n)$  (see [33] for the case  $d > k$ ).

Suppose that  $d = k-1$ . By Lemma 3.7, the homomorphism  $p_j = p_{(j; \emptyset)}$  sends the relation (3.4) to  $\sum_{z \in B_{k-1}(n)} \gamma_{(j; \emptyset), z} z \equiv 0$ . This relation implies  $\gamma_{(j; \emptyset), z} = 0$  for any  $1 \leq j \leq k$  and  $z \in B_{k-1}(n)$ .

Suppose  $0 < \ell(J) < k-3$  and  $\gamma_{(i; I), z} = 0$  for all  $(i; I) \in \mathcal{N}_k$  with  $\ell(I) < \ell(J)$ ,  $1 \leq i \leq k$  and  $z \in B_{k-1}(n)$ . Then, using Lemma 3.7 and the relation (3.3), we see that the homomorphism  $p_{(j; J)}$  sends the relation (3.4) to  $\sum_{z \in B_{k-1}(n)} \gamma_{(j; J), z} z \equiv 0$ . Hence, we get  $\gamma_{(j; J), z} = 0$  for all  $z \in B_{k-1}(n)$ .

Now, let  $(j; J) \in \mathcal{N}_k$  with  $\ell(J) = k-3$ . If  $J \neq I_3$ , then using Lemma 3.7, we have  $p_{(j; J)}(\phi_{(i; I)}(z)) \equiv 0$  for all  $z \in B_{k-1}(n)$  and  $(i; I) \in \mathcal{N}_k$  with  $(i; I) \neq (j; J)$ . So, we get

$$p_{(j; J)}(\mathcal{S}) \equiv \sum_{z \in B_{k-1}(n)} \gamma_{(j; J), z} z \equiv 0.$$

Hence,  $\gamma_{(j; J), z} = 0$ , for all  $z \in B_{k-1}(n)$ .

According to Lemma 3.7,  $p_{(j; I_3)}(\phi_{(1; I_1)}(z)) \equiv 0$  for  $z \in \mathcal{C}$  and  $p_{(j; I_3)}(\phi_{(1; I_1)}(z)) \in \langle \mathcal{E} \rangle$  for  $z \in \mathcal{D} \cup \mathcal{E}$ . Hence, we obtain

$$p_{(j; I_3)}(\mathcal{S}) \equiv \sum_{z \in \mathcal{C} \cup \mathcal{D}} \gamma_{(j; I_3), z} z \equiv 0 \pmod{\langle \mathcal{E} \rangle}.$$

So, we get  $\gamma_{(j; I_3), z} = 0$  for all  $z \in \mathcal{C} \cup \mathcal{D}$ .

Now, let  $(j; J) \in \mathcal{N}_k$  with  $\ell(J) = k-2$ . Suppose that  $I_3 \not\subset J$ . Then, using Lemma 3.7, we have  $p_{(j; J)}(\phi_{(1; I_1)}(z)) \equiv 0$  for all  $z \in \mathcal{B}$ . Hence, we get

$$p_{(j; J)}(\mathcal{S}) \equiv \sum_{z \in \mathcal{B}} \gamma_{(j; J), z} z \equiv 0.$$

From this, we obtain  $\gamma_{(j; J), z} = 0$  for all  $z \in \mathcal{B}$ .

Suppose that  $I_3 \subset J$ . Then, either  $J = I_2, j = 1, 2$  or  $J = I_3 \cup 2, j = 1$ . According to Lemma 3.7,  $p_{(j; I_2)}(\phi_{(1; I_1)}(z)) \in \langle \mathcal{D} \cup \mathcal{E} \rangle$  for all  $z \in \mathcal{B}$ ,  $p_{(j; I_3 \cup 2)}(\phi_{(1; I_1)}(z)) \equiv 0$  for

$z \in \mathcal{C} \cup \mathcal{D}$  and  $p_{(1;I_3 \cup 2)}(\phi_{(1;I_1)}(z)) \in \langle \mathcal{E} \rangle$  for  $z \in \mathcal{E}$ . Hence, we obtain

$$\begin{aligned} p_{(j;I_2)}(\mathcal{S}) &\equiv \sum_{z \in \mathcal{C}} \gamma_{(j;I_2),z} z \equiv 0 \pmod{\langle \mathcal{D} \cup \mathcal{E} \rangle}, \\ p_{(1;I_3 \cup 2)}(\mathcal{S}) &\equiv \sum_{z \in \mathcal{C} \cup \mathcal{D}} \gamma_{(1;I_3 \cup 2),z} z \equiv 0 \pmod{\langle \mathcal{E} \rangle}. \end{aligned}$$

So,  $\gamma_{(j;I_2),z} = 0$  for  $z \in \mathcal{C}$  and  $\gamma_{(1;I_3 \cup 2),z} = 0$  for  $z \in \mathcal{C} \cup \mathcal{D}$ . Since  $\gamma_{(i;I),z} = 0$ , for all  $z \in \mathcal{C}$  and  $I \neq I_1$ , applying Lemma 3.7, we have

$$p_{(1;I_1)}(\mathcal{S}) \equiv \sum_{z \in \mathcal{C}} \gamma_{(1;I_1),z} z \equiv 0 \pmod{\langle \mathcal{D} \cup \mathcal{E} \rangle}.$$

Hence,  $\gamma_{(1;I_1),z} = 0$  for all  $z \in \mathcal{C}$ . So, the relation (3.4) becomes

$$\begin{aligned} \mathcal{S} &= \sum_{1 \leq i \leq 3, z \in \mathcal{E}} \gamma_{(i;I_3),z} \phi_{(i;I_3)}(z) + \sum_{z \in \mathcal{E}} \gamma_{(1;I_3 \cup 2),z} \phi_{(1;I_3 \cup 2)}(z) \\ &+ \sum_{1 \leq i \leq 2, z \in \mathcal{D} \cup \mathcal{E}} \gamma_{(i;I_2),z} \phi_{(i;I_2)}(z) + \sum_{z \in \mathcal{D} \cup \mathcal{E}} \gamma_{(1;I_1),z} \phi_{(1;I_1)}(z) \equiv 0. \end{aligned} \quad (3.5)$$

Using the relation (3.5) and Lemma 3.7,

$$p_{(i;I_2)}(\mathcal{S}) \equiv \sum_{z \in \mathcal{D}} (\gamma_{(i;I_2),z} + \gamma_{(1;I_1),z}) z \equiv 0 \pmod{\langle \mathcal{E} \rangle}, \quad i = 1, 2.$$

This relation implies  $\gamma_{(1;I_2),z} = \gamma_{(2;I_2),z} = \gamma_{(1;I_1),z}$  for all  $z \in \mathcal{D}$ . On the other hand, using the relation (3.5) and Lemma 3.7, one gets

$$p_{(1;I_1)}(\mathcal{S}) \equiv \sum_{z \in \mathcal{D}} (\gamma_{(1;I_2),z} + \gamma_{(2;I_2),z} + \gamma_{(1;I_1),z}) z \equiv 0 \pmod{\langle \mathcal{E} \rangle}.$$

So,  $\gamma_{(1;I_2),z} + \gamma_{(2;I_2),z} + \gamma_{(1;I_1),z} = 0$ . Hence,  $\gamma_{(1;I_2),z} = \gamma_{(2;I_2),z} = \gamma_{(1;I_1),z} = 0$ , for all  $z \in \mathcal{D}$ . Now, the relation (3.5) becomes

$$\begin{aligned} \mathcal{S} &= \sum_{1 \leq i \leq 3, z \in \mathcal{E}} \gamma_{(i;I_3),z} \phi_{(i;I_3)}(z) + \sum_{z \in \mathcal{E}} \gamma_{(1;I_3 \cup 2),z} \phi_{(1;I_3 \cup 2)}(z) \\ &+ \sum_{1 \leq i \leq 2, z \in \mathcal{E}} \gamma_{(i;I_2),z} \phi_{(i;I_2)}(z) + \sum_{z \in \mathcal{E}} \gamma_{(1;I_1),z} \phi_{(1;I_1)}(z) \equiv 0. \end{aligned} \quad (3.6)$$

Using the relation (3.6) and Lemma 3.7, one gets

$$\begin{aligned} p_{(i;I_3)}(\mathcal{S}) &\equiv \sum_{z \in \mathcal{E}} (\gamma_{(i;I_3),z} + \gamma_{(1;I_1),z}) z \equiv 0, \quad i = 1, 2, 3, \\ p_{(1;I_3 \cup 2)}(\mathcal{S}) &\equiv \sum_{z \in \mathcal{E}} (\gamma_{(1;I_3),z} + \gamma_{(2;I_3),z} + \gamma_{(1;I_3 \cup 2),z} + \gamma_{(1;I_1),z}) z \equiv 0, \\ p_{(j;I_2)}(\mathcal{S}) &\equiv \sum_{z \in \mathcal{E}} (\gamma_{(j;I_3),z} + \gamma_{(3;I_3),z} + \gamma_{(j;I_2),z} + \gamma_{(1;I_1),z}) z \equiv 0, \quad j = 1, 2, \\ p_{(1;I_1)}(\mathcal{S}) &\equiv \sum_{z \in \mathcal{E}} (\gamma_{(1;I_3),z} + \gamma_{(2;I_3),z} + \gamma_{(3;I_3),z} \\ &\quad + \gamma_{(1;I_2),z} + \gamma_{(2;I_2),z} + \gamma_{(1;I_3 \cup 2),z} + \gamma_{(1;I_1),z}) z \equiv 0. \end{aligned}$$

From the above relations, we get

$$\gamma_{(i;I_3),z} = \gamma_{(j;I_2),z} = \gamma_{(1;I_3 \cup 2),z} = \gamma_{(1;I_1),z} = 0$$

for all  $z \in \mathcal{E}$ ,  $i = 1, 2, 3$ ,  $j = 1, 2$ . The proposition is completely proved.  $\square$



Let  $n = \sum_{1 \leq i \leq k-1} (2^{d_i} - 1)$  with  $d_i$  positive integers such that  $d_1 > d_2 > \dots > d_{k-2} \geq d_{k-1} = d > 0$ , and let  $m = \sum_{1 \leq i \leq k-2} (2^{d_i-d} - 1)$ . Set  $p = \min\{k, d\}$  and  $\mathcal{N}_{k,p} = \{(i; I) \in \mathcal{N}_k : \ell(I) < p\}$ . Then, we have  $|\mathcal{N}_{k,p}| = \sum_{1 \leq u \leq p} \binom{k}{u}$ . From the proof of Proposition 3.3, we see that the set  $\left[ \bigcup_{(i; I) \in \mathcal{N}_{k,p}} \phi_{(i; I)}(B_{k-1}(n)) \right]$  is linearly independent in  $QP_k$ . So, one gets the following.

**Corollary 3.1.15** (Mothebe [18]).  $\dim(QP_k)_n \geq \sum_{u=1}^p \binom{k}{u} \dim(QP_{k-1})_m$ .

### 3.2. Proof of Lemmas 3.4 and 3.5.

We need the following lemma.

**Lemma 3.2.1.** *Let  $i, j, d, a, b$  be positive integers such that  $i, j \leq k$ ,  $i \neq j$ , and  $a + b = 2^d - 1$ . Then*

$$x := X_i^a X_j^b \simeq_2 X_i^{2^d-2} X_j.$$

*Proof.* We prove the lemma by induction on  $b$ . If  $b = 1$ , then

$$x = X_i^a X_j^b = X_i^{2^d-2} X_j.$$

So, the lemma holds. Suppose that  $b > 1$  and the lemma holds for  $b - 1$ . Since  $i \neq j$ ,  $X_i^a X_j^b = x_i^b x_j^a X_{\{i,j\}}^{2^d-1}$ .

If  $\alpha_0(b) = 0$ , then  $\alpha_0(a) = \alpha_0(2^d - 1 - b) = 1$ . Using the Cartan formula and the inductive hypothesis, we have

$$x \simeq_0 Sq^1(x_i^{b-1} x_j^a X_{\{i,j\}}^{2^d-1}) + x_i^{b-1} x_j^{a+1} X_{\{i,j\}}^{2^d-1} \simeq_1 X_i^{a+1} X_j^{b-1} \simeq_2 X_i^{2^d-2} X_j.$$

By an argument analogous to the previous one, we see that if  $\alpha_0(b) = 1, \alpha_1(b) = 0$ , then

$$\begin{aligned} x &\simeq_0 Sq^1(x_i^{b-2} x_j^{a+1} X_{\{i,j\}}^{2^d-1}) + Sq^2(x_i^{b-2} x_j^a X_{\{i,j\}}^{2^d-1}) + x_i^{b-1} x_j^{a+1} X_{\{i,j\}}^{2^d-1} \\ &\simeq_2 x_i^{b-1} x_j^{a+1} X_{\{i,j\}}^{2^d-1} = X_i^{a+1} X_j^{b-1} \simeq_2 X_i^{2^d-2} X_j. \end{aligned}$$

If  $\alpha_0(b) = \alpha_1(b) = 1$ , then

$$\begin{aligned} x &\simeq_0 Sq^1(x_i^b x_j^{a-1} X_{\{i,j\}}^{2^d-1}) + Sq^2(x_i^{b-1} x_j^{a-1} X_{\{i,j\}}^{2^d-1}) + x_i^{b-1} x_j^{a+1} X_{\{i,j\}}^{2^d-1} \\ &\simeq_2 x_i^{b-1} x_j^{a+1} X_{\{i,j\}}^{2^d-1} = X_i^{a+1} X_j^{b-1} \simeq_2 X_i^{2^d-2} X_j. \end{aligned}$$

The lemma is proved.  $\square$

**Lemma 3.2.2.** *Let  $(i; I) \in \mathcal{N}_k$  and let  $d, h, u$  be integers such that  $\ell(I) = r < h \leq d$ , and  $1 \leq i < u \leq k$ . Then, we have*

$$Y := \phi_{(i; I)}(X^{2^h-1}) X_u^{2^d-2^h} \simeq_{r+2} \phi_{(i; I \cup u)}(X^{2^d-1}).$$

*Proof.* We prove the lemma by induction on  $r$ . If  $r = 0$ , then using Lemma 3.2.1, we have  $Y = X_i^{2^h-1} X_u^{2^d-2^h} \simeq_2 X_i^{2^d-2} X_u = \phi_{(i; u)}(X^{2^d-1})$ . So, the lemma holds.

Suppose that  $r > 0$ ,  $I = (i_1, i_2, \dots, i_r)$  and the lemma is true for  $r - 1$ . A direct computation, using Lemma 3.2.1 and Proposition 2.5(ii), shows

$$\begin{aligned} Y &= \phi_{(i_1; I \setminus i_1)}(X^{2^r-1}) (X_i^{2^{h-r}-1} X_u^{2^{d-r}-2^{h-r}})^{2^r} \\ &\simeq_{r+2} \phi_{(i_1; I \setminus i_1)}(X^{2^r-1}) (X_u X_i^{2^{d-r}-2})^{2^r} \\ &= \left( \phi_{(i_1; I \setminus i_1)}(X^{2^r-1}) X_u^{2^r} \right) X_i^{2^d-2^{r+1}} := Z. \end{aligned}$$

If  $u < i_1$ , then  $Z = \phi_{(i; I \cup u)}(X^{2^d-1})$ . Suppose  $u \geq i_1$ .

If  $u \notin I$ , then  $u > i_1$ . By the inductive hypothesis and Proposition 2.5(i), we have

$$Z \simeq_{r+1} \phi_{(i_1; I \cup u \setminus i_1)}(X^{2^{r+1}-1})X_i^{2^d-2^{r+1}} = \phi_{(i; I \cup u)}(X^{2^d-1}).$$

If  $u \in I$  and  $r = 1$ , then  $u = i_1$ . Using Lemma 3.2.1, we have

$$Z = X_i^{2^d-2^{r+1}}X_{i_1}^{2^{r+1}-1} \simeq_2 X_i^{2^d-2}X_{i_1} = \phi_{(i; I \cup u)}(X^{2^d-1}).$$

Suppose that  $u \in I$  and  $r > 1$ . If  $u = i_1$ , then  $Z = \phi_{(i_1; I \setminus i_1)}(X^{2^{r+1}-1})X_i^{2^d-2^{r+1}}$ . If  $u > i_1$ , then using the inductive hypothesis and Proposition 2.5(i), we obtain

$$Z \simeq_{r+1} \phi_{(i_1; I \cup u \setminus i_1)}(X^{2^{r+1}-1})X_i^{2^d-2^{r+1}} = \phi_{(i_1; I \setminus i_1)}(X^{2^{r+1}-1})X_i^{2^d-2^{r+1}}.$$

Now, applying Lemma 3.2.1 and Proposition 2.5(ii), one gets

$$\begin{aligned} \phi_{(i_1; I \setminus i_1)}(X^{2^{r+1}-1})X_i^{2^d-2^{r+1}} &= \phi_{(i_2; I \setminus \{i_1, i_2\})}(X^{2^{r-1}-1})(X_{i_1}^3 X_i^{2^d-r+1-4})^{2^{r-1}} \\ &\simeq_{r+1} \phi_{(i_2; I \setminus \{i_1, i_2\})}(X^{2^{r-1}-1})(X_{i_1} X_i^{2^d-r+1-2})^{2^{r-1}} = \phi_{(i; I \cup u)}(X^{2^d-1}). \end{aligned}$$

The above equalities imply  $Y \simeq_{r+2} \phi_{(i; I \cup u)}(X^{2^d-1})$ . The lemma is proved.  $\square$

*Proof of Lemma 3.4.* We prove the lemma by induction on  $d$ . For  $d = 1$ , the lemma holds with  $i = j_0$  and  $I = \emptyset$ .

Let  $d = 2$ . If  $j_0 = j_1 = i$ , then  $x = \phi_{(i, \emptyset)}(X^3)$ . If  $j = j_0 > j_1 = i$ , then  $x = X_i^2 X_j = \phi_{(i, j)}(X^3)$ . If  $i = j_0 < j_1 = j$ , then

$$x = X_i X_j^2 \simeq_0 S q^1(X_\emptyset X_{\{i, j\}}^2) + X_i^2 X_j \simeq_1 X_i^2 X_j = \phi_{(i, j)}(X^3).$$

So, the lemma holds for  $d = 2$ .

Suppose  $d > 2$  and the lemma holds for  $d-1$ . By the inductive hypothesis, there is  $(h; H) \in \mathcal{N}_k$  such that

$$\prod_{0 \leq t < d-1} X_{j_t}^{2^t} \simeq_{d-2} \phi_{(h; H)}(X^{2^{d-1}-1}),$$

where  $h = \min\{j_0, j_1, \dots, j_{d-2}\}$ . If  $j_{d-1} = h$ , then the lemma holds with  $(i; I) = (h; H)$ . Assume that  $j_{d-1} \neq h$ . If  $H = \emptyset$ , then  $j_0 = j_1 = \dots = j_{d-2} = h$ . Using Proposition 2.5(i) and Lemma 3.2.1, we obtain

$$x \simeq_{d-1} X_h^{2^{d-1}-1} X_{j_{d-1}}^{2^{d-1}} \simeq_2 X_i^{2^{d-1}-1} X_{i_1} = \phi_{(i; i_1)}(X^{2^d-1}),$$

where  $i = \min\{h, j_{d-1}\} = \min\{j_0, j_1, \dots, j_{d-1}\}$ ,  $i_1 = \max\{h, j_{d-1}\}$ . So, the lemma is true.

Suppose  $H = (h_1, \dots, h_s)$  with  $\{h_1, \dots, h_s\} = \{j_0, \dots, j_{d-2}\} \setminus \{h\}$  and  $0 < s < d-1$ . Then, we have  $\phi_{(h; H)}(X^{2^{d-1}-1}) = \phi_{(h_1; H \setminus h_1)}(X^{2^s-1})X_h^{2^{d-1}-2^s}$ . If  $h < j_{d-1}$  and  $s < d-2$ , then by Lemma 3.2.2, we have

$$x \simeq_{d-2} \phi_{(h; H)}(X^{2^{d-1}-1})X_{j_{d-1}}^{2^{d-1}} \simeq_{s+2} \phi_{(h; H \cup j_{d-1})}(X^{2^d-1}).$$

Since  $s+2 \leq d-1$ , the lemma is true with  $i = h$  and  $I = H \cup j_{d-1}$ .

Suppose that  $h < j_{d-1}$  and  $s = d - 2$ . From the proof of Lemma 3.2.1, we have  $X_h X_{j_{d-1}}^2 \simeq_1 X_h^2 X_{j_{d-1}}$ . Hence, using Proposition 2.5, one gets

$$\begin{aligned} x &\simeq_{d-2} \phi_{(h_1; H \setminus h_1)}(X^{2^{d-2}-1})(X_h X_{j_{d-1}}^2)^{2^{d-2}} \\ &\simeq_{d-1} \phi_{(h_1; H \setminus h_1)}(X^{2^{d-2}-1})(X_h^2 X_{j_{d-1}})^{2^{d-2}} \\ &= \phi_{(h_1; H \setminus h_1)}(X^{2^{d-2}-1}) X_{j_{d-1}}^{2^{d-2}} X_h^{2^{d-1}}. \end{aligned}$$

By the inductive hypothesis, we have

$$\phi_{(h_1; H \setminus h_1)}(X^{2^{d-2}-1}) X_{j_{d-1}}^{2^{d-2}} \simeq_{d-2} \phi_{(j; J)}(X^{2^{d-1}-1}),$$

where  $j = \min\{h_1, \dots, h_s, j_{d-1}\} = \min(\{j_0, \dots, j_{d-1}\} \setminus \{h\})$ ,  $J = (H \cup j_{d-1}) \setminus j$ .

If  $j_{d-1} \notin H$ , then from the above equalities and Proposition 2.5(i), we get

$$x \simeq_{d-1} \phi_{(j; J)}(X^{2^{d-1}-1}) X_h^{2^{d-1}} = \phi_{(h; J \cup j)}(X^{2^d-1}).$$

The lemma holds with  $i = h$ ,  $I = J \cup j = H \cup j_{d-1}$ .

If  $j_{d-1} \in H$ , then  $\ell(J) = d - 3$ . By Lemma 3.2.1,  $X_j^3 X_h^4 \simeq_2 X_j X_h^6$ . Using Proposition 2.5, we obtain

$$\begin{aligned} x &\simeq_{d-1} \phi_{(j; J)}(X^{2^{d-1}-1}) X_h^{2^{d-1}} = \phi_{(j_1; J \setminus j_1)}(X^{2^{d-3}-1})(X_j^3 X_h^4)^{2^{d-3}} \\ &\simeq_{d-1} \phi_{(j_1; J \setminus j_1)}(X^{2^{d-3}-1})(X_j X_h^6)^{2^{d-3}} = \phi_{(h; J \cup j)}(X^{2^d-1}). \end{aligned}$$

Here  $j_1 = \min J$ . The lemma holds with  $i = h$  and  $I = J \cup j = H = H \cup j_{d-1}$ .

If  $h > j_{d-1}$  and  $s = d - 2$ , then using Proposition 2.5(i), we have

$$x \simeq_{d-1} \phi_{(h; H)}(X^{2^{d-1}-1}) X_{j_{d-1}}^{2^{d-1}} = \phi_{(j_{d-1}; H \cup h)}(X^{2^d-1}).$$

Suppose that  $h > j_{d-1}$  and  $s < d - 2$ . By Lemma 3.2.1,  $X_h^{2^{d-s-1}-1} X_{j_{d-1}}^{2^{d-s-1}} \simeq_2 X_h X_{j_{d-1}}^{2^{d-s}-2}$ . Since  $s + 2 < d$ , using Proposition 2.5, we obtain

$$\begin{aligned} x &\simeq_{d-1} \phi_{(h_1; H \setminus h_1)}(X^{2^s-1})(X_h^{2^{d-s-1}-1} X_{j_{d-1}}^{2^{d-s-1}})^{2^s} \\ &\simeq_{d-1} \phi_{(h_1; H \setminus h_1)}(X^{2^s-1})(X_h X_{j_{d-1}}^{2^{d-s}-2})^{2^s} = \phi_{(j_{d-1}; H \cup h)}(X^{2^d-1}). \end{aligned}$$

The lemma holds with  $i = j_{d-1}$ ,  $I = H \cup h$ . □

*Proof of Lemma 3.5.* Applying the Cartan formula, we have

$$Sq^1(X_\emptyset^{2^c-1} y_0^{2^c}) = \sum_{1 \leq j \leq k} X_j^{2^c-1} y_j^{2^c},$$

where  $c$  is a positive integer. From this, we obtain

$$X_i^{2^c-1} y_i^{2^c} \equiv \sum_{1 \leq j < i} X_j^{2^c-1} y_j^{2^c} + \sum_{i < j \leq k} X_j^{2^c-1} y_j^{2^c}. \quad (3.7)$$

If  $r = 0$ , then  $t_j = j$  and  $I^{(j)} = \emptyset$  for  $j > i$ . Then, the first part of the lemma follows from the relation (3.7) with  $c = d$ .

If  $d > r > 0$ , then  $\phi_{(i;I)}(X^{2^d-1})y_i^{2^d} = \phi_{(i_1;I \setminus i_1)}(X^{2^r-1})(X_i^{2^c-1}y_i^{2^c})^{2^r}$ , with  $c = d - r > 0$  and  $i_1 = \min I$ . Hence, using Lemma 2.14 and the relation (3.7), we get

$$\begin{aligned} \phi_{(i;I)}(X^{2^d-1})y_i^{2^d} &\equiv \sum_{1 \leq j < i} \phi_{(i_1;I \setminus i_1)}(X^{2^r-1})(X_j^{2^c-1}y_j^{2^c})^{2^r} \\ &\quad + \sum_{i < j \leq k} \phi_{(i_1;I \setminus i_1)}(X^{2^r-1})(X_j^{2^c-1}y_j^{2^c})^{2^r}. \end{aligned}$$

A simple computation, using Lemmas 3.2.2 and 2.14, shows

$$\begin{aligned} \phi_{(i_1;I \setminus i_1)}(X^{2^r-1})(X_j^{2^c-1}y_j^{2^c})^{2^r} &= \phi_{(j;I)}(X^{2^d-1})y_j^{2^d}, \text{ for } j < i, \\ \phi_{(i_1;I \setminus i_1)}(X^{2^r-1})(X_j^{2^c-1}y_j^{2^c})^{2^r} &\equiv \phi_{(t_j;I^{(j)})}(X^{2^d-1})y_j^{2^d}, \text{ for } j > i. \end{aligned}$$

Hence, the first part of the lemma follows.

If  $d > r + 1$ , then  $\phi_{(i;I)}(X^{2^d-1})y_i^{2^d} = \phi_{(i;I)}(X^{2^{r+1}-1})(X_i^{2^c-1}y_i^{2^c})^{2^{r+1}}$ , with  $c = d - r - 1 > 0$ . Hence, using the relation (3.7) and Lemma 2.14, we get

$$\begin{aligned} \phi_{(i;I)}(X^{2^d-1})y_i^{2^d} &\equiv \sum_{1 \leq j < i} \phi_{(i;I)}(X^{2^{r+1}-1})(X_j^{2^c-1}y_j^{2^c})^{2^{r+1}} \\ &\quad + \sum_{i < j \leq k} \phi_{(i;I)}(X^{2^{r+1}-1})(X_j^{2^c-1}y_j^{2^c})^{2^{r+1}}. \end{aligned}$$

Applying Lemmas 3.2.2 and 2.14, we have

$$\begin{aligned} \phi_{(i;I)}(X^{2^{r+1}-1})(X_j^{2^c-1}y_j^{2^c})^{2^{r+1}} &= \phi_{(j;I \cup i)}(X^{2^d-1})y_j^{2^d}, \text{ for } j < i, \\ \phi_{(i;I)}(X^{2^{r+1}-1})(X_j^{2^c-1}y_j^{2^c})^{2^{r+1}} &\equiv \phi_{(i;I \cup j)}(X^{2^d-1})y_j^{2^d}, \text{ for } j > i. \end{aligned}$$

So, the second part of the lemma is proved.  $\square$

### 3.3. Proof of Lemma 3.6.

In this subsection, we denote  $\omega_{(t,k,d)} = \omega((x_{t+1} \dots x_k)^{2^d-1}x_t^{2^d})$  with  $d \geq k - t + 1$ ,  $P_{(t,k)} = \mathbb{F}_2[x_t, \dots, x_k] \subset P_{(1,k)} = P_k$ , and  $n_t = \deg(\omega_{(t,k,d)}) = (k - t)(2^d - 1) + 2^d$ .

**Lemma 3.3.1.** *Let  $x$  be a monomial of degree  $n_t$  in  $P_{(t,k)}$ . If  $x \notin P_{(t,k)}^-(\omega_{(t,k,d)})$ , then  $\omega(x) = \omega_{(t,k,d)}$ .*

The proof of the lemma is similar to the one of Lemma 2.13 (see the proof of Lemma 2.14 in [33] for  $t = 1$ ).

Denote by  $R_{(t,k,d)}$  the subspace of  $P_{(t,k)}$  spanned by all  $Sq^{2^i}(y)$  with  $0 \leq i \leq k - t$ , where  $y$  is a monomial of degree  $n_t - 2^i$  in  $P_{(t,k)}$  such that  $\omega_j(y) > 0$  for some  $j > d$ .

For  $J = (j_1, j_2, \dots, j_i)$ ,  $t \leq j_1 < j_2 < \dots < j_i \leq k$ ,  $0 \leq i \leq k - t + 1$ , we set

$$y_{(J,t,k,d)} = (x_t x_{t+1} \dots x_k)^{2^d-1} / (x_{j_1}^{2^{i-1}} x_{j_2}^{2^{i-2}} \dots x_{j_i}^{2^0}).$$

Here, by convention,  $x_{j_1}^{2^{i-1}} x_{j_2}^{2^{i-2}} \dots x_{j_i}^{2^0} = 1$  for  $i = 0$ . Denote

$$Y_{(t,k,d)} = \sum_{u=t}^k x_t^{2^{k-t}-1} y_{(I_t, t+1, k, d)} x_u^{2^d}.$$

**Lemma 3.3.2.** *For  $d \geq k \geq 2$ ,  $\omega = \omega_{(1,k,d)}$ , and  $J^* = (1, 2, \dots, k - 1)$ ,*

$$\phi_{(1;I_1)}(X^{2^d-1})x_k^{2^d} \simeq_{(0,\omega)} Y_{(1,k-1,d)}x_k^{2^d-1} + Sq^{2^{k-1}}(y_{(J^*,1,k,d)}) \bmod(R_{(1,k,d)}).$$

*Proof.* We observe that  $\phi_{(1;I_1)}(X^{2^d-1})x_k^{2^d} = (\phi_{(1;I_1)}(X^{2^k-1})x_k^{2^k})(X_1x_k)^{2^d-2^k}$  and  $\phi_{(1;I_1)}(X^{2^k-1})x_k^{2^k} = \left(\prod_{j=1}^k X_j^{2^{k-j}}\right)x_k^{2^k}$ . By a direct computation using the Cartan formula, we get

$$\phi_{(1;I_1)}(X^{2^k-1})x_k^{2^k} \simeq_{(0,\omega_{(1,k,k)})} \sum_{u=1}^{k-1} \left(\prod_{j=1}^{k-1} X_j^{2^{k-j-1}}\right) X_u^{2^{k-1}} x_u^{2^k} + \sum_{i=0}^{k-1} Sq^{2^i}(W_i),$$

where  $W_i = (\prod_{j=1}^{k-i-1} X_j^{2^{k-j}})(\prod_{j=k-i}^{k-1} X_j^{2^{k-j-1}})X_\emptyset^{2^i}x_k^{2^{k-2^{i+1}}}$ . Using Lemma 3.4, we have  $\left(\prod_{j=1}^{k-1} X_j^{2^{k-j-1}}\right)X_u^{2^{k-1}} \simeq_{k-1} \phi_{(1;I_1^*)}(X^{2^k-1})$ , for  $1 < u < k$  and  $I_1^* = (2, 3, \dots, k-1)$ . Hence,

$$\left(\prod_{j=1}^{k-1} X_j^{2^{k-j-1}}\right)X_u^{2^{k-1}} \simeq_{(0,\omega_{(1,k,k)})} \phi_{(1;I_1^*)}(X^{2^k-1}) + \sum_{j=0}^{k-2} Sq^{2^j}(g_j),$$

where  $g_j$  are suitable polynomials in  $P_k$ .

A simple computation shows that  $\left(\prod_{j=1}^{k-1} X_j^{2^{k-j-1}}\right)X_1^{2^{k-1}} = \phi_{(1;I_1^*)}(X^{2^k-1})$ ; if  $g \in P_k^-(\omega_{(1,k,k)})$ , then  $(X_1x_k)^{2^d-2^k}g \in P_k^-(\omega)$ ;  $W_{k-1} = y_{(J^*,1,k,k)}$ ;

$$Sq^{2^i}(W_i)(X_1x_k)^{2^d-2^k} = Sq^{2^i}(W_i(X_1x_k)^{2^d-2^k}) \in R_{(1,k,d)}, 0 \leq i \leq k-2;$$

$$Sq^{2^j}(g_j)x_u^{2^k}(X_1x_k)^{2^d-2^k} = Sq^{2^j}(g_jx_u^{2^k}(X_1x_k)^{2^d-2^k}) \in R_{(1,k,d)}, 0 \leq j \leq k-2.$$

From the above equalities, we obtain

$$\begin{aligned} \phi_{(1;I_1)}(X^{2^d-1})x_k^{2^d} &\simeq_{(0,\omega)} \sum_{u=1}^{k-1} \phi_{(1;I_1^*)}(X^{2^k-1})x_u^{2^k}(X_1x_k)^{2^d-2^k} \\ &\quad + Sq^{2^{k-1}}(y_{(J^*,1,k,k)})(X_1x_k)^{2^d-2^k} \bmod(R_{(1,k,d)}), \end{aligned}$$

Since  $\sum_{u=1}^{k-1} \phi_{(1;I_1^*)}(X^{2^k-1})x_u^{2^k} = Y_{(1,k-1,k)}x_k^{2^{k-1}}$ , the lemma is true for  $d = k$ .

Suppose that  $d > k$ . Then,  $Sq^{2^{k-1}}(y_{(J^*,1,k,k)})(X_1x_k)^{2^d-2^k} \in R_{(1,k,d)}$ . According to Lemma 3.3.1,  $\phi_{(1;I_1^*)}(X^{2^k-1})x_u^{2^k}(X_1x_k)^{2^d-2^k} \in P_k^-(\omega)$ , for  $1 < u < k$ .

For  $u = 1$ , we have

$$\phi_{(1;I_1^*)}(X^{2^k-1})x_1^{2^k}(X_1x_k)^{2^d-2^k} = \phi_{(2;I_2^*)}(X^{2^{k-2}-1})(X_1^{2^{d-k+2}-3}X_k^2)^{2^{k-2}}x_k^{2^d}.$$

Here  $I_2^* = (3, \dots, k-1)$ . By Lemma 3.2.1,  $X_1^{2^{d-k+2}-3}X_k^2 \simeq_2 X_1X_k^{2^{d-k+2}-2}$ . So, using Proposition 2.5, we obtain

$$\begin{aligned} \phi_{(2;I_2^*)}(X^{2^{k-2}-1})(X_1^{2^{d-k+2}-3}X_k^2)^{2^{k-2}} &\simeq_k \phi_{(2;I_2^*)}(X^{2^{k-2}-1})(X_1X_k^{2^{d-k+2}-2})^{2^{k-2}} \\ &= \phi_{(1;I_1^*)}(X^{2^{k-1}-1})X_k^{2^d-2^{k-1}}. \end{aligned}$$

This equality implies

$$\begin{aligned} \phi_{(1;I_1^*)}(X^{2^k-1})x_1^{2^k}(X_1x_k)^{2^d-2^k} &\simeq_{(0,\omega)} \\ &\quad \left(\phi_{(1;I_1^*)}(X^{2^{k-1}-1})X_k^{2^d-2^{k-1}} + \sum_{i=0}^{k-1} Sq^{2^i}(p_i)\right)x_k^{2^d}, \end{aligned}$$

where  $p_i$  are suitable polynomials in  $P_k$ . It is easy to see that  $Sq^{2^i}(p_i)x_k^{2^d} = Sq^{2^i}(p_i x_k^{2^d}) \in R_{(1,k,d)}$ . So, combining the above equalities gives

$$\phi_{(1;I_1)}(X^{2^d-1})x_k^{2^d} \simeq_{(0,\omega)} \phi_{(1;I_1^*)}(X^{2^{k-1}-1})X_k^{2^d-2^{k-1}}x_k^{2^d} \pmod{R_{(1,k,d)}}.$$

Using the Cartan formula, we have

$$\begin{aligned} \phi_{(1;I_1^*)}(X^{2^{k-1}-1})X_k^{2^d-2^{k-1}}x_k^{2^d} &\simeq_{(0,\omega)} Sq^{2^{k-1}}\left(\phi_{(1;I_1^*)}(X^{2^{k-1}-1})X_\emptyset^{2^d-2^{k-1}}\right) \\ &\quad + \sum_{u=1}^{k-1} \phi_{(1;I_1^*)}(X^{2^{k-1}-1})X_u^{2^d-2^{k-1}}x_u^{2^d}. \end{aligned}$$

A simple computation shows:  $\phi_{(1;I_1^*)}(X^{2^{k-1}-1})X_\emptyset^{2^d-2^{k-1}} = y_{(J^*,1,k,d)}$ ;

$$\begin{aligned} \phi_{(1;I_1^*)}(X^{2^{k-1}-1})X_u^{2^d-2^{k-1}} &= \phi_{(1;I_1^*)}(X_{\{k-1,k\}}^{2^{k-1}-1})X_{\{u,k\}}^{2^d-2^{k-1}}x_k^{2^d-1}, \\ \phi_{(1;I_1^*)}(X_{\{k-1,k\}}^{2^{k-1}-1})X_{\{1,k\}}^{2^d-2^{k-1}} &= \phi_{(1;I_1^*)}(X_{\{k-1,k\}}^{2^d-1}). \end{aligned}$$

For  $1 < u < k$ ,  $I_1^* \cup u = I_1^*$ . Since  $\ell(I_1^*) = k-2$ , applying Lemma 3.2.2 for  $P_{k-1}$ ,

$$\phi_{(1;I_1^*)}(X_{\{k-1,k\}}^{2^{k-1}-1})X_{\{u,k\}}^{2^d-2^{k-1}} \simeq_k \phi_{(1;I_1^*)}(X_{\{k-1,k\}}^{2^d-1}).$$

This equality implies

$$\phi_{(1;I_1^*)}(X^{2^{k-1}-1})X_u^{2^d-2^{k-1}}x_u^{2^d} \simeq_{(0,\omega)} \left(\phi_{(1;I_1^*)}(X_{\{k-1,k\}}^{2^d-1}) + \sum_{i=0}^{k-1} Sq^{2^i}(h_i)\right)x_u^{2^d}x_k^{2^d-1},$$

where  $h_i$  are suitable polynomials in  $P_{k-1}$ . Using the Cartan formula and Lemma 3.3.1, we obtain  $Sq^{2^i}(h_i)x_u^{2^d}x_k^{2^d-1} \in R_{(1,k,d)} + P_k^-(\omega)$ .

Combining the above equalities gives

$$\begin{aligned} \phi_{(1;I_1)}(X^{2^d-1})x_k^{2^d} &\simeq_{(0,\omega)} \phi_{(1;I_1^*)}(X^{2^{k-1}-1})X_k^{2^d-2^{k-1}}x_k^{2^d} \pmod{R_{(1,k,d)}} \\ &\simeq_{(0,\omega)} \left(\sum_{u=1}^{k-1} \phi_{(1;I_1^*)}(X_{\{k-1,k\}}^{2^d-1})x_u^{2^d}\right)x_k^{2^d-1} + Sq^{2^{k-1}}(y_{(J^*,1,k,d)}) \pmod{R_{(1,k,d)}} \\ &= Y_{(1,k-1,d)}x_k^{2^d-1} + Sq^{2^{k-1}}(y_{(J^*,1,k,d)}) \pmod{R_{(1,k,d)}}. \end{aligned}$$

The lemma is proved.  $\square$

Set  $\mathcal{M}_{(t,k,i)} = \{J = (j_1, j_2, \dots, j_i) : t \leq j_1 < j_2 < \dots < j_i \leq k\}, 0 \leq i \leq k-t+1$ .

**Lemma 3.3.3.** For  $d \geq k-t+1$ ,

$$Y_{(t,k,d)} \simeq_{(0,\omega_{(t,k,d)})} \sum_{0 \leq i \leq k-t} \sum_{J \in \mathcal{M}_{(t,k,i)}} Sq^{2^i}(y_{(J,t,k,d)}) \pmod{R_{(t,k,d)}}.$$

*Proof.* Note that  $Y_{(t,k,d)} = Y_1(x_t, \dots, x_k) \in P_{(t,k)}$ . So, we need only to prove the lemma for  $Y_1 = Y_{(1,k,d)}$  and  $\omega = \omega_{(1,k,d)} = \omega(X_1^{2^d-1}x_1^{2^d})$ . The proof proceeds by induction on  $k$ .

It is easy to see that  $Y_{(1,1,d)} = x_1^{2^d} = Sq^1(x_1^{2^d-1})$ ,  $y_{(\emptyset,1,1,d)} = x_1^{2^d-1}$ ,  $\mathcal{M}_{(1,1,0)} = \{\emptyset\}$  and  $R_{(1,1,d)} = 0$ . Hence, the lemma is true for  $k=1$ .

Suppose that  $k > 1$  and the lemma is true for  $k-1$ . We have

$$Y_{(1,k,d)} = X_k Y_{(1,k-1,d-1)}^2 x_k^{2^d-2} + \phi_{(1;I_1)}(X^{2^d-1})x_k^{2^d}.$$

It is easy to see that if  $h \in P_{k-1}^-(\omega_{(1,k-1,d-1)})$ , then  $X_k x_k^{2^d-2} h^2 \in P_k^-(\omega)$ . So, using the inductive hypothesis, we get

$$\begin{aligned} X_k Y_{(1,k-1,d-1)}^2 x_k^{2^d-2} &\simeq_{(0,\omega)} \sum_{j=0}^{k-2} X_k (Sq^{2^j}(h_j))^2 x_k^{2^d-2} \\ &+ \sum_{0 \leq i \leq k-2} \sum_{J \in \mathcal{M}_{(1,k-1,i)}} X_k (Sq^{2^i}(y_{(J,1,k-1,d-1)}))^2 x_k^{2^d-2}, \end{aligned}$$

where  $h_j$  are suitable polynomials in  $P_{k-1}$ .

By a direct computation using Proposition 2.4, Lemma 3.3.1 and the Cartan formula, we obtain  $X_k (Sq^{2^j}(h_j))^2 x_k^{2^d-2} \in R_{(1,k,d)} + P_k^-(\omega)$ , and

$$\begin{aligned} X_k (Sq^{2^i}(y_{(J,1,k-1,d-1)}))^2 x_k^{2^d-2} &\simeq_{(0,\omega)} \\ Sq^{2^{i+1}}(X_k y_{(J,1,k-1,d-1)}^2 x_k^{2^d-2}) &+ (X_k y_{(J,1,k-1,d-1)}^2 x_k^{2^d-2}) x_k^{2^{i+1}}. \end{aligned}$$

A simple computation shows that  $X_k y_{(J,1,k-1,d-1)}^2 x_k^{2^d-2} = y_{(J \cup k, 1, k, d)}$ .

Observe that if  $g \in P_{k-1}^-(\omega_{(1,k-1,d)})$ , then  $g x_k^{2^d-1} \in P_k^-(\omega)$ . Hence, using Lemma 3.3.2 and the inductive hypothesis, we have

$$\begin{aligned} \phi_{(1;I_1)}(X^{2^d-1}) x_k^{2^d} &\simeq_{(0,\omega)} \sum_{j=0}^{k-2} Sq^{2^j}(g_j) x_k^{2^d-1} + Sq^{2^{k-1}}(y_{(J^*, 1, k, d)}) \\ &+ \sum_{0 \leq i \leq k-2} \sum_{J \in \mathcal{M}_{(1,k-1,i)}} Sq^{2^i}(y_{(J,1,k-1,d)}) x_k^{2^d-1}, \end{aligned}$$

where  $g_j$  are suitable polynomials in  $P_{k-1}$ . Using Lemma 3.3.1 and the Cartan formula, we get  $Sq^{2^j}(g_j) x_k^{2^d-1} \in R_{(1,k,d)} + P_k^-(\omega)$  and

$$Sq^{2^i}(y_{(J,1,k-1,d)}) x_k^{2^d-1} \simeq_{(0,\omega)} Sq^{2^i}(y_{(J,1,k,d)}) + y_{(J,1,k,d)} x_k^{2^i}.$$

If  $J = \emptyset$ , then  $i = 0$  and  $y_{(J,1,k,d)} x_k^{2^i} = y_{(J \cup k, 1, k, d)} x_k^{2^{i+1}} = X_k^{2^d-1} x_k^{2^d}$ . If  $J \neq \emptyset$ , then

$$\begin{aligned} y_{(J,1,k,d)} x_k^{2^i} &= \phi_{(j_1; J \setminus j_1)}(X^{2^i-1}) X_k^{2^d-2^i} x_k^{2^d} \\ y_{(J \cup k, 1, k, d)} x_k^{2^{i+1}} &= \phi_{(j_1; J \cup k \setminus j_1)}(X^{2^{i+1}-1}) X_k^{2^d-2^{i+1}} x_k^{2^d}. \end{aligned}$$

Here  $j_1 = \min J$ . Since  $0 \leq \ell(J \setminus j_1) < \ell(J \cup k \setminus j_1) = i \leq k-2$ , using Lemma 3.2.2, we obtain

$$\begin{aligned} y_{(J,1,k,d)} x_k^{2^i} &= \phi_{(j_1; J \setminus j_1)}(X^{2^i-1}) X_k^{2^d-2^i} \simeq_k \phi_{(j_1; J \cup k \setminus j_1)}(X^{2^d-1}), \\ y_{(J \cup k, 1, k, d)} x_k^{2^{i+1}} &= \phi_{(j_1; J \cup k \setminus j_1)}(X^{2^{i+1}-1}) X_k^{2^d-2^{i+1}} \simeq_k \phi_{(j_1; J \cup k \setminus j_1)}(X^{2^d-1}). \end{aligned}$$

This implies

$$y_{(J,1,k,d)} x_k^{2^i} + y_{(J \cup k, 1, k, d)} x_k^{2^{i+1}} \simeq_{(0,\omega)} \sum_{j=0}^{k-1} Sq^{2^j}(z_j) x_k^{2^d},$$

where  $z_j$  are suitable polynomials in  $P_k$ . By using the Cartan formula, we have  $Sq^{2^j}(z_j) x_k^{2^d} = Sq^{2^j}(z_j x_k^{2^d}) \in R_{(1,k,d)}$ . It is easy to see that  $\mathcal{M}_{(1,k,0)} = \mathcal{M}_{(1,k-1,0)} = \{\emptyset\}$ ,  $\mathcal{M}_{(1,k-1,k-1)} = \{J^*\}$  and

$$\mathcal{M}_{(1,k,i)} = \{J \cup k : J \in \mathcal{M}_{(1,k-1,i-1)}\} \cup \mathcal{M}_{(1,k-1,i)}, \quad 1 \leq i \leq k-1.$$

Now, the lemma follows from the above equalities.  $\square$

*Proof of Lemma 3.6.* For  $t = 1$ , the lemma follows from Lemma 3.3.3. For  $t > 1$ , we have  $Y_t = Z^{2^d-1}Y_{(t,k,d)}$  with  $Z = x_1x_2 \dots x_{t-1}$ .

Observe that if  $h \in P_{(t,k)}^-(\omega_{(t,k,d)})$ , then  $Z^{2^d-1}h \in P_k^-(\omega)$ .

Let  $Sq^{2^i}(y) \in R_{(t,k,d)}$  with  $y$  a monomial in  $P_{(t,k)}$ ,  $0 \leq i \leq k-t$ , and  $g := Z^{2^d-1}Sq^{2^i}(y)$ . According to the Cartan formula,

$$g = Sq^{2^i}(Z^{2^d-1}y) + \sum_{1 \leq v \leq 2^i} Sq^v(Z^{2^d-1})Sq^{2^i-v}(y).$$

For  $1 \leq v < 2^i$ , by using the Cartan formula, we can easily show that if a monomial  $z$  appears as a term of the polynomial  $Sq^v(Z^{2^d-1})Sq^{2^i-v}(y)$ , then  $\omega_u(z) < k-1$  for some  $u \leq d$ . So,  $Sq^v(Z^{2^d-1})Sq^{2^i-v}(y) \in P_k^-(\omega)$ . Hence, using the Cartan formula and Lemma 3.3.1, we get

$$g \simeq_{(i+1,\omega)} Sq^{2^i}(Z^{2^d-1}y) \simeq_{(0,\omega)} \sum_{1 \leq j < t} Z^{2^d-1}x_j^{2^i}y \in P_k^-(\omega).$$

Let  $f := Z^{2^d-1}Sq^{2^i}(y_{(J,t,k,d)})$  with  $J \in \mathcal{M}_{(t,k,i)}$ ,  $0 \leq i \leq k-t$ . By an argument analogous to the previous one, we have

$$f \simeq_{(i+1,\omega)} \sum_{1 \leq j < t} Z^{2^d-1}x_j^{2^i}y_{(J,t,k,d)} = \sum_{1 \leq j < t} \phi_{(j;J)}(X^{2^d-1})x_j^{2^d}.$$

Since  $i+1 \leq k-t+1$ , using Lemma 3.3.3 and the above equalities, we obtain

$$Y_t = Z^{2^d-1}Y_{(t,k,d)} \simeq_{(k-t+1,\omega)} \sum_{1 \leq j < t} \sum_{J \in \mathcal{M}_{(t,k)}} \phi_{(j;J)}(X^{2^d-1})x_j^{2^d}.$$

Here  $\mathcal{M}_{(t,k)} = \bigcup_{i=0}^{k-t} \mathcal{M}_{(t,k,i)}$ . Obviously  $J \subset I_{t-1}$  and  $J \neq I_{t-1}$  for all  $J \in \mathcal{M}_{(t,k)}$ . The lemma follows.  $\square$

### 3.4. Proof of Lemma 3.7.

First, we prepare some lemmas.

**Lemma 3.4.1.** *Let  $d$  be a positive integer,  $\omega = \omega(X^{2^d-1})$  and  $(j; J), (i; I) \in \mathcal{N}_k$  with  $\ell(I) < d$ . Then*

$$p_{(j;J)}\phi_{(i;I)}(X^{2^d-1}) \simeq_{(0,\omega)} \begin{cases} X^{2^d-1}, & \text{if } (i; I) \subset (j; J), \\ 0, & \text{if } (i; I) \not\subset (j; J). \end{cases}$$

*Proof.* Suppose that  $(i; I) \not\subset (j; J)$ . If  $i \notin (j; J)$ , then from (3.1), we see that  $p_{(j;J)}(\phi_{(i;I)}(X^{2^d-1}))$  is a sum of monomials of the form

$$w = x_{i'}^{2^r-1}f_{k-1;i'}(z),$$

for suitable monomial  $z$  in  $P_{k-2}$ . Here  $i' = i$  if  $j > i$  and  $i' = i-1$  if  $j < i$ . In this case, we have  $\alpha_r(2^r-1) = 0$  and  $\omega_{r+1}(w) < k-1$ . Hence,  $w \in P_{k-1}^-(\omega)$ . Suppose that  $i \in (j; J)$ . Since  $(i; I) \not\subset (j; J)$ , there is  $1 \leq t \leq r$ , such that  $i_t \notin (j; J)$ , then from (3.1), we see that  $p_{(j;J)}(\phi_{(i;I)}(X^{2^d-1}))$  is a sum of monomials of the form

$$w = x_{i_t-1}^{2^r-2^{r-t}-1}f_{k-1;i_t-1}(z),$$



for some monomial  $z$  in  $P_{k-2}$ . It is easy to see that  $\alpha_{r-t}(2^r - 2^{r-t} - 1) = 0$  and  $\omega_{r-t+1}(w) < k - 1$ . Hence,  $w \in P_{k-1}^-(\omega)$ .

Suppose that  $(i; I) \subset (j; J)$ . If  $i = j$ , then from (3.1), we see that the polynomial  $p_{(j; J)}(\phi_{(i; I)}(X^{2^d-1}))$  is a sum of monomials of the form

$$w = \left( \prod_{1 \leq t \leq r} x_{i_t-1}^{2^r-2^{r-t}-1+b_t} \right) \left( \prod_{j+1 \in J \setminus I} x_j^{2^d-1+c_j} \right) \left( \prod_{j+1 \notin J} x_j^{2^d-1} \right),$$

where  $b_1 + b_2 + \dots + b_r + \sum_{j+1 \in J \setminus I} c_j = 2^r - 1$ . If  $c_j > 0$ , then  $\alpha_{u_j}(2^d - 1 + c_j) = 0$  with  $u_j$  the smallest index such that  $\alpha_{u_j}(c_j) = 1$ . Hence,  $w \in P_{k-1}^-(\omega)$ . If  $b_t = 0$  for suitable  $1 \leq t \leq r$ , then  $\alpha_{r-t}(2^r - 2^{r-t} - 1) = 0$  and  $\omega_{r-t+1}(w) < k - 1$ . Hence,  $w \in P_{k-1}^-(\omega)$ . Suppose that  $b_t > 0$  for any  $t$ . Let  $v_t$  be the smallest index such that  $\alpha_{v_t}(b_t) = 1$ . If  $v_t \neq r - t$ , then  $\alpha_{v_t}(2^r - 2^{r-t} - 1 + b_t) = 0$  and  $w \in P_{k-1}^-(\omega)$ . So,  $u_t = r - t$  and  $b_t = 2^{r-t} + b'_t$  with  $b'_t \geq 0$ . If  $b'_t > 0$ , then  $\alpha_{v'_t}(2^r - 2^{r-t} - 1 + b_t) = \alpha_{v'_t}(2^r - 1 + b'_t) = 0$  with  $v'_t$  the smallest index such that  $\alpha_{v'_t}(b'_t) = 1$ . Hence,  $w \in P_{k-1}^-(\omega)$ . If  $b'_t = 0$  for  $1 \leq t \leq r$ , then  $w = X^{2^d-1}$ .

If  $i \in J$ , then from (3.1), we see that the polynomial  $p_{(j; J)}(\phi_{(i; I)}(X^{2^d-1}))$  is a sum of monomials of the form

$$w = x_{i-1}^{2^r-1+b_0} \left( \prod_{1 \leq t \leq r} x_{i_t-1}^{2^r-2^{r-t}-1+b_t} \right) \left( \prod_{j+1 \in J \setminus (i; I)} x_j^{2^d-1+c_j} \right) \left( \prod_{j+1 \notin J} x_j^{2^d-1} \right),$$

where  $b_0 + b_1 + b_2 + \dots + b_r + \sum_{j+1 \in J \setminus (i; I)} c_j = 2^d - 1$ . By a same argument as above, we see that  $w \in P_{k-1}^-(\omega)$  if either  $c_j > 0$  or  $b_t \neq 2^{r-t}$  for some  $j, t$  with  $t > 0$ . Suppose  $c_j = 0$  and  $b_t = 2^{r-t}$  with all  $j$  and  $t > 0$ . Then,  $2^d - 1 = b_0 + b_1 + b_2 + \dots + b_r + \sum_{j+1 \in J \setminus (i; I)} c_j = b_0 + 2^r - 1$  and  $w = X^{2^d-1}$ . The lemma is proved.  $\square$

The following is easily proved by a direct computation.

**Lemma 3.4.2.** *The following diagram is commutative:*

$$\begin{array}{ccc} P_{k-1} & \xrightarrow{f_{k,i}} & P_k \\ \downarrow p_{(i; I_i^*)} & & \downarrow p_{(i+1; I_{i+1})} \\ P_{k-2} & \xrightarrow{f_{k-1,i}} & P_{k-1}. \end{array}$$

Here  $I_i^* = (i + 1, \dots, k - 1)$  for  $1 \leq i < k$ .

*Proof of Lemma 3.7.*

i) Suppose that either  $d \geq k$  or  $d = k - 1$  and  $I \neq I_1$ , then  $\phi_{(i; I)}(z) = \phi_{(i; I)}(X^{2^d-1})f_i(\bar{z})^{2^d}$ . Hence, the first part of the lemma follows from Lemmas 3.4.1 and 2.14.

ii) Let  $z \in \mathcal{C}$  and  $d = k - 1$ . According to the relation (3.3),  $\phi_{(1; I_1)}(z) = \phi_{(2; I_2)}(X^{2^d-1})f_{k,1}(\bar{z})^{2^d}$ . Hence, from Lemmas 3.4.1, 3.4.2 and 2.14, we have

$$\begin{aligned} p_{(i; I)}(\phi_{(1; I_1)}(z)) &\equiv p_{(i; I)}(\phi_{(2; I_2)}(X^{2^d-1}))p_{(i; I)}(f_{k,1}(\bar{z})^{2^d}) \\ &\equiv \begin{cases} z & \text{if } (i; I) = (1; I_1), \\ X^{2^d-1}f_{k-1,1}(p_{(1; I_1^*)}(\bar{z}^{2^d})) \in \langle \mathcal{D} \cup \mathcal{E} \rangle, & \text{if } (i; I) = (2; I_2), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

iii) Let  $z \in \mathcal{D}$  and  $d = k - 1$ . Since  $\nu_1(z) = 2^{k-1} - 1$ , we have  $\nu_1(\bar{z}) = 0$ . Hence, using the relation (3.3), Lemmas 3.4.1, 3.4.2 and 2.14, we get

$$\begin{aligned} p_{(i;I)}(\phi_{(1;I_1)}(z)) &\equiv p_{(i;I)}(\phi_{(3;I_3)}(X^{2^d-1}))p_{(i;I)}(f_{k,2}(\bar{z})^{2^d}) \\ &\equiv \begin{cases} z & \text{if } I_2 \subset I, \\ X^{2^d-1}f_{k-1,2}(p_{(2;I_2^*)}(\bar{z})^{2^d}) \in \langle \mathcal{E} \rangle, & \text{if } (i;I) = (3;I_3), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

iv) Let  $z \in \mathcal{E}$  and  $d = k - 1$ . Using the relation (3.3), Lemmas 3.4.1 and 2.14, we have  $p_{(i;I)}(\phi_{(1;I_1)}(z)) \equiv 0$ , if  $(4;I_4) \not\subset (i;I)$ . Suppose  $(4;I_4) \subset (i;I)$ . If  $I_3 = (4;I_4) \not\subset I$ , then  $(i;I) = (4;I_4)$ . Using Lemmas 3.4.1, 3.4.2 and 2.14, one gets

$$\begin{aligned} p_{(4;I_4)}(\phi_{(1;I_1)}(z)) &= p_{(4;I_4)}(\phi_{(4;I_4)}(X^{2^d-1}))p_{(4;I_4)}(f_{k,3}(\bar{z})^{2^d}) \\ &\equiv X^{2^d-1}f_{k-1,3}(p_{(3;I_3^*)}(\bar{z}^{2^d})). \end{aligned}$$

Suppose that the monomial  $y$  is a term of  $f_{k-1,3}(p_{(3;I_3^*)}(\bar{z}^{2^d}))$ . Since  $\nu_1(\bar{z}) = \nu_2(\bar{z}) = 0$ , we have  $\omega_1(y) < k - 3$ . According to Theorem 2.12,  $X^{2^d-1}y \equiv 0$ . Hence,  $X^{2^d-1}f_{k-1,3}(p_{(3;I_3^*)}(\bar{z}^{2^d})) \equiv 0$ . Now, using Lemmas 3.4.1, 3.4.2 and 2.14, we get

$$p_{(i;I)}(\phi_{(1;I_1)}(z)) \equiv p_{(i;I)}(\phi_{(4;I_4)}(X^{2^d-1}))p_{(i;I)}(f_3(\bar{z})^{2^d}) \equiv \begin{cases} z & \text{if } I_3 \subset I, \\ 0, & \text{otherwise.} \end{cases}$$

The lemma is completely proved.  $\square$

#### 4. THE CASES $k \leq 3$

In this section and the next sections, we denote by  $B_k(n)$  the set of all admissible monomials of degree  $n$  in  $P_k$ . For a subset  $B \subset P_k$ , we denote  $B^0 = B \cap P_k^0$ ,  $B^+ = B \cap P_k^+$ . For an  $\omega$ -vector  $\omega = (\omega_1, \omega_2, \dots, \omega_m)$  of degree  $n$ , we set  $B_k(\omega) = B_k(n) \cap P_k(\omega)$ .

If there is  $i_0 = 0, i_1, i_2, \dots, i_r > 0$  such that  $i_1 + i_2 + \dots + i_r = m$  and  $\omega_{i_1+\dots+i_{s-1}+t} = a_s, 1 \leq t \leq i_s, 1 \leq s \leq r$ , then we denote  $\omega = (a_1^{(i_1)}, a_2^{(i_2)}, \dots, a_r^{(i_r)})$ . If  $i_u = 1$ , then we denote  $a_u^{(1)} = a_u$ .

Using Lemma 5.3.3(i) in Subsection 5.3 and Theorem 2.9, we easily obtain the following.

**Proposition 4.1.** *For any  $s \geq 1$ ,*

$$B_k(1^{(s)}) = \{x_{i_1}x_{i_2}^2 \dots x_{i_{m-1}}^{2^{m-2}}x_{i_m}^{2^s-2^{m-1}}; 1 \leq i_1 < \dots < i_m \leq k, 1 \leq m \leq \min\{s, k\}\}.$$

It is well known that if  $n \neq 2^u - 1$  then  $B_1(n) = \emptyset$ . If  $n = 2^u - 1$  for  $u \geq 0$ , then  $B_1(n) = B_1(1^{(u)}) = \{x^{2^u-1}\}$ . It is easy to see that  $\Phi(B_1(0)) = \{1\} = B_2(0)$ ,  $\Phi(B_1(1)) = \{x_1, x_2\} = B_2(1)$ . According to a result in Peterson [22], for  $u > 1$ , we have

$$B_2(2^u - 1) = \Phi(B_1(2^u - 1)) = \{x_1^{2^u-1}, x_2^{2^u-1}, x_1x_2^{2^u-2}\},$$

By Theorem 1.1,  $B_2(n) = \emptyset$  if  $n \neq 2^{t+u} + 2^t - 2$  for all nonnegative integers  $t, u$ . We define the  $\mathbb{F}_2$ -linear map  $\psi : (P_k)_m \rightarrow (P_k)_{2m+k}$  by  $\psi(y) = X_\emptyset y^2$  for any monomial  $y \in (P_k)_m$ . From Theorem 1.2, we have

**Theorem 4.2** (Peterson [22]). *If  $n = 2^{t+u} + 2^t - 2$ , with  $t, u$  positive integers, then*

$$B_2(n) = \psi^t(\Phi(B_1(2^u - 1)))$$

$$= \begin{cases} \{(x_1 x_2)^{2^t - 1}\}, & u = 0, \\ \{x_1^{2^{t+1}-1} x_2^{2^t-1}, x_1^{2^t-1} x_2^{2^{t+1}-1}\}, & u = 1, \\ \{x_1^{2^{t+u}-1} x_2^{2^t-1}, x_1^{2^t-1} x_2^{2^{t+u}-1}, x_1^{2^{t+1}-1} x_2^{2^{t+u}-2^t-1}\}, & u > 1. \end{cases}$$

By Theorems 1.1 and 1.2, for  $k = 3$ , we need only to consider the cases of degree  $n = 2^s - 2$ ,  $n = 2^s - 1$  and  $n = 2^{s+t} + 2^s - 2$  with  $s, t$  positive integers. By a direct computation using Theorem 1.3 we have

**Theorem 4.3** (Kameko [14]).

- i) *If  $n = 2^s - 2$ , then  $B_3(2^s - 2) = \Phi(B_2(2^s - 2))$ .*
- ii) *If  $n = 2^s - 1$ , then  $B_3(2^s - 1) = B_3(1^{(s)}) \cup \psi(\Phi(B_2(2^{s-1} - 2)))$ .*
- iii) *If  $n = 2^{s+t} + 2^s - 2$ , then*

$$B_3(n) = \begin{cases} \Phi(B_2(8)) \cup \{x_1^3 x_2^4 x_3\}, & \text{if } s = 1, t = 2, \\ \Phi(B_2(2^{s+t} + 2^s - 2)), & \text{otherwise.} \end{cases}$$

## 5. PROOF OF THEOREM 1.4

For  $1 \leq i \leq k$ , define  $\varphi_i : QP_k \rightarrow QP_k$ , the homomorphism induced by the  $\mathcal{A}$ -homomorphism  $\overline{\varphi}_i : P_k \rightarrow P_k$ , which is determined by  $\overline{\varphi}_1(x_1) = x_1 + x_2$ ,  $\overline{\varphi}_1(x_j) = x_j$  for  $j > 1$ , and  $\overline{\varphi}_i(x_i) = x_{i-1}$ ,  $\overline{\varphi}_i(x_{i-1}) = x_i$ ,  $\overline{\varphi}_i(x_j) = x_j$  for  $j \neq i, i-1$ ,  $1 < i \leq k$ . Note that the general linear group  $GL_k$  is generated by  $\overline{\varphi}_i$ ,  $0 < i \leq k$  and the symmetric group  $\Sigma_k$  is generated by  $\overline{\varphi}_i$ ,  $1 < i \leq k$ .

Let  $B$  be a finite subset of  $P_k$  consisting of some monomials of degree  $n$ . To prove the set  $[B]$  is linearly independent in  $QP_k$ , we order the set  $B$  by the order as in Definition 2.6 and denote the elements of  $B$  by  $d_i = d_{n,i}$ ,  $0 < i \leq b = |B|$  in such a way that  $d_{n,i} < d_{n,j}$  if and only if  $i < j$ . Suppose there is a linear relation

$$\mathcal{S} = \sum_{1 \leq j \leq b} \gamma_j d_{n,j} \equiv 0,$$

with  $\gamma_j \in \mathbb{F}_2$ . For  $(i; I) \in \mathcal{N}_k$ , we explicitly compute  $p_{(i;I)}(\mathcal{S})$  in terms of a minimal set of  $\mathcal{A}$ -generators in  $P_{k-1}$ . Computing from some relations  $p_{(i;I)}(\mathcal{S}) \equiv 0$  with  $(i; I) \in \mathcal{N}_k$  and  $\overline{\varphi}_i(\mathcal{S}) \equiv 0$ , we will obtain  $\gamma_j = 0$  for all  $j$ .

### 5.1. The case of degree $n = 2^{s+1} - 3$ .

In this subsection we prove the following.

**Proposition 5.1.1.** *For any  $s \geq 1$ ,  $\Phi(B_3(2^{s+1} - 3))$  is a minimal set of generators for  $\mathcal{A}$ -module  $P_4$  in degree  $2^{s+1} - 3$ .*

We can see that  $\Phi(B_3(2^{s+1} - 3))$  is the set of all the admissible monomials of degree  $n = 2^{s+1} - 3$ ,  $|\Phi(B_3(1))| = 4$ ,  $|\Phi(B_3(5))| = 15$ ,  $|\Phi(B_3(13))| = 35$  and  $|\Phi(B_3(2^{s+1} - 3))| = 45$ , for  $s \geq 4$ . We need the following lemmas for the proof of the proposition.

**Lemma 5.1.2.** *If  $x$  is an admissible monomial of degree  $2^{s+1} - 3$  in  $P_4$ , then  $\omega(x) = (3^{(s-1)}, 1)$ .*

*Proof.* It is easy to see that the lemma holds for  $s = 1$ . Suppose  $s \geq 2$ . Obviously,  $z = x_1^{2^s-1}x_2^{2^{s-1}-1}x_3^{2^{s-1}-1}$  is the minimal spike of degree  $2^{s+1} - 3$  in  $P_4$  and  $\omega(z) = (3^{(s-1)}, 1)$ . Since  $2^{s+1} - 3$  is odd, we get either  $\omega_1(x) = 1$  or  $\omega_1(x) = 3$ . If  $\omega_1(x) = 1$ , then  $\omega(x) < \omega(z)$ . By Theorem 2.12,  $x$  is hit. This contradicts the fact that  $x$  is admissible. Hence, we have  $\omega_1(x) = 3$ . Using Proposition 2.10 and Theorem 2.12, we obtain  $\omega_i(x) = 3$ ,  $i = 1, 2, \dots, s-1$ . Since  $\deg x = 2^{s+2} - 3$ , from this it implies  $\omega_s(x) = 1$  and  $\omega_i(x) = 0$  for  $i > s$ . The lemma is proved.  $\square$

By a direct computation, we have the following.

**Lemma 5.1.3.** *The following monomials are strictly inadmissible:*

$$X_1x_1^2, X_iX_j^2, \quad 1 \leq i < j \leq 4.$$

*Proof of Proposition 5.1.1.* We have  $n = 2^{s+1} - 3 = 2^s + 2^{s-1} + 2^{s-1} - 3$ . Hence, the proposition follows from Theorem 1.3 for  $s \geq 4$ . According to Theorem 4.3,

$$B_3(n) = \{v_1 = X^{2^{s-1}-1}x_3^{2^{s-1}}, v_2 = X^{2^{s-1}-1}x_2^{2^{s-1}}, v_3 = X^{2^{s-1}-1}x_1^{2^{s-1}}\},$$

where  $X = x_1x_2x_3$ .

It is easy to see that  $\Phi(B_3(1)) = \{x_1, x_2, x_3, x_4\}$ . Hence, the proposition holds for  $s = 1$ . For  $s = 2$ , using Lemma 5.1.3, we see that

$$\Phi^+(B_3(5)) = \{x_1x_2x_3x_4^2, x_1x_2x_3^2x_4, x_1x_2^2x_3x_4\}$$

is a minimal set of generators for  $(P_4^+)_5$ . Using Theorem 4.3, it is easy to see that, for  $s = 3$ ,  $\Phi^+(B_3(13))$  is the set of 23 monomials:

$$\begin{aligned} d_1 &= x_1x_2^2x_3^3x_4^7, & d_2 &= x_1x_2^2x_3^7x_4^3, & d_3 &= x_1x_2^3x_3^2x_4^7, & d_4 &= x_1x_2^3x_3^3x_4^6, \\ d_5 &= x_1x_2^3x_3^6x_4^3, & d_6 &= x_1x_2^3x_3^7x_4^2, & d_7 &= x_1x_2^6x_3^3x_4^3, & d_8 &= x_1x_2^7x_3^2x_4^3, \\ d_9 &= x_1x_2^7x_3^3x_4^2, & d_{10} &= x_1^3x_2^2x_3^5x_4^7, & d_{11} &= x_1^3x_2^2x_3^3x_4^6, & d_{12} &= x_1^3x_2^2x_3^6x_4^3, \\ d_{13} &= x_1^3x_2^2x_3^7x_4^2, & d_{14} &= x_1^3x_2^3x_3^6x_4^4, & d_{15} &= x_1^3x_2^3x_3^3x_4^4, & d_{16} &= x_1^3x_2^3x_3^4x_4^3, \\ d_{17} &= x_1^3x_2^3x_3^5x_4^2, & d_{18} &= x_1^3x_2^5x_3^2x_4^3, & d_{19} &= x_1^3x_2^5x_3^3x_4^2, & d_{20} &= x_1^3x_2^7x_3^2x_4^2, \\ d_{21} &= x_1^7x_2x_3^2x_4^3, & d_{22} &= x_1^7x_2x_3^3x_4^2, & d_{23} &= x_1^7x_2^3x_3x_4^2. \end{aligned}$$

By a direct computation using Lemmas 5.1.2, 5.1.3 and Theorem 2.12, if  $x$  is an admissible monomial of degree 13 in  $P_4^+$ , then  $x \in \Phi^+(B_3(13))$ . Hence,  $(QP_4^+)_{13}$  is spanned by  $[\Phi^+(B_3(13))]$ .

Now we prove that the set  $[\Phi^+(B_3(13))]$  is linearly independent. Suppose there is a linear relation

$$\mathcal{S} = \sum_{j=1}^{23} \gamma_j d_j \equiv 0, \quad (5.1.3.1)$$

where  $\gamma_j \in \mathbb{F}_2, 1 \leq j \leq 23$ .

Consider the homomorphisms  $p_{(1;i)} : P_4 \rightarrow P_3, i = 2, 3, 4$ . By a direct computation from (5.1.3.1), we have

$$\begin{aligned} p_{(1;2)}(\mathcal{S}) &\equiv \gamma_1v_1 + \gamma_2v_2 + \gamma_7v_3 \equiv 0, \\ p_{(1;3)}(\mathcal{S}) &\equiv \gamma_3v_1 + (\gamma_5 + \gamma_{16})v_2 + \gamma_8v_3 \equiv 0, \\ p_{(1;4)}(\mathcal{S}) &\equiv (\gamma_4 + \gamma_{15})v_1 + \gamma_6v_2 + \gamma_9v_3 \equiv 0. \end{aligned}$$

From the above equalities it implies

$$\begin{cases} \gamma_j = 0, & j = 1, 2, 3, 6, 7, 8, 9, \\ \gamma_5 = \gamma_{16}, & \gamma_4 = \gamma_{15}. \end{cases} \quad (5.1.3.2)$$

Substituting (5.1.3.2) into the relation (5.1.3.1), we have

$$\mathcal{S} = \gamma_4 d_4 + \gamma_5 d_5 + \sum_{10 \leq j \leq 23} \gamma_j d_j \equiv 0. \quad (5.1.3.3)$$

Applying the homomorphisms  $p_{(2;3)}, p_{(2;4)}, p_{(3;4)} : P_4 \rightarrow P_3$  to (5.1.3.3), we get

$$\begin{aligned} p_{(2;3)}(\mathcal{S}) &\equiv \gamma_{10} v_1 + (\gamma_{12} + \gamma_{16} + \gamma_{18}) v_2 + \gamma_{21} v_3 \equiv 0, \\ p_{(2;4)}(\mathcal{S}) &\equiv (\gamma_{11} + \gamma_{15} + \gamma_{19}) v_1 + \gamma_{13} v_2 + \gamma_{22} v_3 \equiv 0, \\ p_{(3;4)}(\mathcal{S}) &\equiv (\gamma_{14} + \gamma_{15} + \gamma_{16} + \gamma_{17}) v_1 + \gamma_{20} v_2 + \gamma_{23} v_3 \equiv 0. \end{aligned}$$

Hence, we get

$$\begin{cases} \gamma_j = 0, & j = 10, 13, 20, 21, 22, 23, \\ \gamma_{12} + \gamma_{16} + \gamma_{18} = \gamma_{11} + \gamma_{15} + \gamma_{19} = 0, \\ \gamma_{14} + \gamma_{15} + \gamma_{16} + \gamma_{17} = 0. \end{cases} \quad (5.1.3.4)$$

Substituting (5.1.3.4) into the relation (5.1.3.3) we get

$$\mathcal{S} = \gamma_4 d_4 + \gamma_5 d_5 + \gamma_{11} d_{11} + \gamma_{12} d_{12} + \sum_{14 \leq j \leq 19} \gamma_j d_j \equiv 0. \quad (5.1.3.5)$$

The homomorphisms  $p_{(1;(2,3))}, p_{(1;(2,4))}, p_{(1;(3,4))} : P_4 \rightarrow P_3$ , send (5.1.3.5) respectively to

$$\begin{aligned} p_{(1;(2,3))}(\mathcal{S}) &\equiv (\gamma_5 + \gamma_{12} + \gamma_{16}) v_2 + \gamma_{18} v_3 \equiv 0, \\ p_{(1;(2,4))}(\mathcal{S}) &\equiv (\gamma_4 + \gamma_{11} + \gamma_{15}) v_1 + \gamma_{19} v_3 \equiv 0 \\ p_{(1;(3,4))}(\mathcal{S}) &\equiv (\gamma_4 + \gamma_{14} + \gamma_{15}) v_1 + (\gamma_5 + \gamma_{16} + \gamma_{17}) v_2 \equiv 0. \end{aligned}$$

From this we obtain

$$\begin{cases} \gamma_{18} = \gamma_{19} = \gamma_5 + \gamma_{12} + \gamma_{16} = 0, \\ \gamma_4 + \gamma_{11} + \gamma_{15} = \gamma_4 + \gamma_{14} + \gamma_{15} = \gamma_5 + \gamma_{16} + \gamma_{17} = 0. \end{cases} \quad (5.1.3.6)$$

Combining (5.1.3.2), (5.1.3.4) and (5.1.3.6), we obtain  $\gamma_j = 0$ ,  $j = 1, 2, \dots, 23$ . The proposition is proved.  $\square$

## 5.2. The case of degree $n = 2^{s+1} - 2$ .

It is well-known that Kameko's homomorphism

$$\widetilde{Sq}_*^0 : (QP_k)_{2m+k} \rightarrow (QP_k)_m$$

is an epimorphism. Hence, we have

$$(QP_k)_{2m+k} \cong (QP_k)_m \oplus (QP_k^0)_{2m+k} \oplus (\text{Ker} \widetilde{Sq}_*^0 \cap (QP_k^+)_{2m+k}),$$

and  $(QP_k)_m \cong \langle [\psi(B_k(m))] \rangle \subset (QP_k)_{2m+k}$ .

For  $k = 4$ , from Theorem 4.3, it is easy to see that

$$\Phi(B_3(2)) = \Phi^0(B_3(2)) = \{x_i x_j : 1 \leq i < j \leq 4\}.$$

For  $m = 2^s - 3$ ,  $s \geq 2$ , we have

$$|\Phi^0(B_3(6))| = 18, \quad |\Phi^0(B_3(2^{s+1} - 2))| = 22, \quad \text{for } s \geq 3,$$

$$|\psi(B_4(1))| = 4, \quad \text{Ker} \widetilde{Sq}_*^0 \cap [B_4^+(6)] = \{[x_1 x_2^2 x_3 x_4^2], [x_1 x_2 x_3^2 x_4^2]\}.$$

Hence,  $\dim(QP_4)_2 = 6$ ,  $\dim(QP_4)_6 = 24$ .

The main result of this subsection is:

**Proposition 5.2.1.** *For any  $s \geq 3$ ,  $(QP_4^+)_{2^{s+1}-2} \cap \text{Ker} \widetilde{Sq}_*^0$  is an  $\mathbb{F}_2$ -vector space of dimension 13 with a basis consisting of all the classes represented by the following admissible monomials:*

$$\begin{aligned} d_1 &= x_1 x_2 x_3^{2^s-2} x_4^{2^s-2}, & d_2 &= x_1 x_2^2 x_3^{2^s-4} x_4^{2^s-1}, & d_3 &= x_1 x_2^2 x_3^{2^s-3} x_4^{2^s-2}, \\ d_4 &= x_1 x_2^2 x_3^{2^s-1} x_4^{2^s-4}, & d_5 &= x_1 x_2^3 x_3^{2^s-4} x_4^{2^s-2}, & d_6 &= x_1 x_2^3 x_3^{2^s-2} x_4^{2^s-4}, \\ d_7 &= x_1 x_2^{2^s-2} x_3 x_4^{2^s-2}, & d_8 &= x_1 x_2^{2^s-1} x_3^2 x_4^{2^s-4}, & d_9 &= x_1^3 x_2 x_3^{2^s-4} x_4^{2^s-2}, \\ d_{10} &= x_1^3 x_2 x_3^{2^s-2} x_4^{2^s-4}, & d_{12} &= x_1^3 x_2^{2^s-3} x_3^2 x_4^{2^s-4}, & d_{13} &= x_1^{2^s-1} x_2 x_3^2 x_4^{2^s-4}. \\ d_{11} &= x_1^3 x_2^3 x_3^4 x_4^4, \text{ for } s = 3, & d_{11} &= x_1^3 x_2^5 x_3^{2^s-6} x_4^{2^s-4}, \text{ for } s > 3, \end{aligned}$$

The proof of this proposition is based on some lemmas.

**Lemma 5.2.2.** *If  $x$  is an admissible monomial of degree  $2^{s+1} - 2$  in  $P_4$  and  $[x] \in \text{Ker} \widetilde{Sq}_*^0$ , then  $\omega(x) = (2^{(s)})$ .*

*Proof.* We prove the lemma by induction on  $s$ . Obviously, the lemma holds for  $s = 1$ . Observe that  $z = (x_1 x_2)^{2^s-1}$  is the minimal spike of degree  $2^{s+1} - 2$  in  $P_4$  and  $\omega(z) = (2^{(s)})$ . Since  $2^{s+1} - 2$  is even, using Theorem 2.12 and the fact that  $[x] \in \text{Ker} \widetilde{Sq}_*^0$ , we obtain  $\omega_1(x) = 2$ . Hence,  $x = x_i x_j y^2$ , where  $y$  is a monomial of degree  $2^s - 2$  and  $1 \leq i < j \leq 4$ . Since  $x$  is admissible, by Theorem 2.9,  $y$  is also admissible. Now, the lemma follows from the inductive hypothesis.  $\square$

The following lemma is proved by a direct computation.

**Lemma 5.2.3.** *The following monomials are strictly inadmissible:*

- i)  $x_i^2 x_j x_k^3, x_i^3 x_j^4 x_k^7, i < j, k \neq i, j, x_1^2 x_2^2 x_3 x_4, x_1^2 x_2 x_3^2 x_4, x_1^2 x_2 x_3 x_4^2, x_1 x_2^2 x_3^2 x_4.$
- ii)  $x_1 x_2^6 x_3^3 x_4^4, x_1^3 x_2^4 x_3 x_4^6, x_1^3 x_2^4 x_3^3 x_4^4.$
- iii)  $x_1 x_2^7 x_3^{10} x_4^{12}, x_1^7 x_2 x_3^{10} x_4^{12}, x_1^3 x_2^3 x_3^{12} x_4^{12}, x_1^3 x_2^5 x_3^8 x_4^{14}, x_1^3 x_2^5 x_3^{14} x_4^8, x_1^7 x_2^7 x_3^8 x_4^8.$

*Proof of Proposition 5.2.1.* Let  $x$  be an admissible monomial in  $P_4$  and  $[x] \in \text{Ker} \widetilde{Sq}_*^0$ . By Lemma 5.2.2,  $\omega_i(x) = 2$ , for  $1 \leq i \leq s$ . By induction on  $s$ , we see that if  $x \neq d_i$ , for  $i = 1, 2, \dots, 13$ , then there is a monomial  $w$ , which is given in Lemma 5.2.3 such that  $x = wy^{2^u}$  for some monomial  $y$  and positive integer  $u$ . By Theorem 2.9,  $x$  is inadmissible. Hence,  $\text{Ker} \widetilde{Sq}_*^0 \cap (QP_4^+)$  is spanned by the classes  $[d_i]$  with  $i = 1, 2, \dots, 13$ .

Now, we prove that the classes  $[d_i]$  with  $i = 1, 2, \dots, 13$ , are linearly independent. Suppose there is a linear relation

$$\mathcal{S} = \sum_{1 \leq i \leq 13} \gamma_i d_i \equiv 0, \quad (5.2.3.1)$$

with  $\gamma_i \in \mathbb{F}_2$ . According to Theorem 4.3, for  $s \geq 3$ ,  $B_3(n) \cap (P_3^+)_n$  is the set consisting of 4 monomials:

$$\begin{aligned} w_1 &= x_1 x_2^{2^s-2} x_3^{2^s-1}, & w_2 &= x_1 x_2^{2^s-1} x_3^{2^s-2}, \\ w_3 &= x_1^3 x_2^{2^s-3} x_3^{2^s-2}, & w_4 &= x_1^{2^s-1} x_2 x_3^{2^s-2}. \end{aligned}$$

Apply the homomorphisms  $p_{(1;2)}, p_{(3;4)} : P_4 \rightarrow P_3$  to the relation (5.2.3.1) to obtain

$$\begin{aligned} p_{(1;2)}(\mathcal{S}) &\equiv \gamma_2 w_1 + \gamma_4 w_2 + \gamma_3 w_3 + \gamma_7 w_4 \equiv 0. \\ p_{(3;4)}(\mathcal{S}) &\equiv \gamma_7 w_1 + \gamma_8 w_2 + \gamma_{12} w_3 + \gamma_{13} w_4 \equiv 0. \end{aligned}$$

From these relations, we get  $\gamma_i = 0$ ,  $i = 2, 3, 4, 7, 8, 12, 13$ . Then, the relation (5.2.3.1) becomes

$$\mathcal{S} = \gamma_1 d_1 + \gamma_5 d_5 + \gamma_6 d_6 + \gamma_9 d_9 + \gamma_{10} d_{10} + \gamma_{11} d_{11} \equiv 0. \quad (5.2.3.2)$$

Apply the homomorphisms  $p_{(1;4)}, p_{(2;3)} : P_4 \rightarrow P_3$  to the relation (5.2.3.2) to get

$$\begin{aligned} p_{(1;4)}(\mathcal{S}) &\equiv (\gamma_1 + \gamma_5 + \gamma_{10} + \gamma_{11})w_1 + \gamma_6 w_3 \equiv 0, \\ p_{(2;3)}(\mathcal{S}) &\equiv (\gamma_1 + \gamma_5 + \gamma_{10} + \gamma_{11})w_2 + \gamma_9 w_3 \equiv 0. \end{aligned}$$

These equalities imply  $\gamma_6 = \gamma_9 = \gamma_1 + \gamma_5 + \gamma_{10} + \gamma_{11} = 0$ . Hence, we obtain

$$\mathcal{S} = \gamma_1 d_1 + \gamma_5 d_5 + \gamma_{10} d_{10} + \gamma_{11} d_{11} \equiv 0. \quad (5.2.3.3)$$

For  $s > 3$ , applying the homomorphisms  $p_{(1;3)}, p_{(2;4)} : P_4 \rightarrow P_3$  to (5.2.3.3), we get

$$\begin{aligned} p_{(1;3)}(\mathcal{S}) &\equiv \gamma_1 w_2 + \gamma_5 w_3 \equiv 0, \\ p_{(2;4)}(\mathcal{S}) &\equiv \gamma_1 w_1 + \gamma_{10} w_3 \equiv 0. \end{aligned}$$

From the above equalities, we get  $\gamma_i = 0, i = 1, 2, \dots, 13$ .

For  $s = 3$ , applying the homomorphisms  $p_{(1;3)}, p_{(2;4)} : P_4 \rightarrow P_3$  to (5.2.3.3), we get

$$\begin{aligned} p_{(1;3)}(\mathcal{S}) &\equiv (\gamma_1 + \gamma_{11})w_2 + \gamma_8 w_3 \equiv 0, \\ p_{(2;4)}(\mathcal{S}) &\equiv (\gamma_1 + \gamma_{11})w_1 + \gamma_{10} w_3 \equiv 0. \end{aligned}$$

From the above equalities, we get  $\gamma_i = 0, i = 2, \dots, 10, 12, 13$  and  $\gamma_1 = \gamma_{11}$ . So, the relation (5.2.3.3) becomes

$$\gamma_1(d_1 + d_{11}) \equiv 0.$$

Now, we prove that  $[d_1 + d_{11}] \neq 0$ . Suppose the contrary, that the polynomial  $d_1 + d_{11} = x_1 x_2 x_3^6 x_4^6 + x_1^3 x_2^3 x_3^4 x_4^4$  is hit. Then, by the unstable property of the action of  $\mathcal{A}$  on the polynomial algebra, we have

$$x_1 x_2 x_3^6 x_4^6 + x_1^3 x_2^3 x_3^4 x_4^4 = Sq^1(A) + Sq^2(B) + Sq^4(C),$$

for some polynomials  $A \in (P_4^+)_{13}, B \in (P_4^+)_{12}, C \in (P_4^+)_{10}$ . Let  $(Sq^2)^3$  act on the both sides of the above equality. Since  $(Sq^2)^3 Sq^1 = 0$  and  $(Sq^2)^3 Sq^2 = 0$ , we get

$$(Sq^2)^3(x_1 x_2 x_3^6 x_4^6 + x_1^3 x_2^3 x_3^4 x_4^4) = (Sq^2)^3 Sq^4(C).$$

On the other hand, by a direct computation, it is not difficult to check that

$$(Sq^2)^3(x_1 x_2 x_3^6 x_4^6 + x_1^3 x_2^3 x_3^4 x_4^4) \neq (Sq^2)^3 Sq^4(C),$$

for all  $C \in (P_4^+)_{10}$ . This is a contradiction. Hence,  $[d_1 + d_{11}] \neq 0$  and  $\gamma_1 = \gamma_{11} = 0$ . The proposition is proved.  $\square$

### 5.3. The case of degree $n = 2^{s+1} - 1$ .

First, we determine the  $\omega$ -vector of an admissible monomial of degree  $2^{s+1} - 1$  in  $P_4$ .

**Lemma 5.3.1.** *If  $x$  is an admissible monomial of degree  $2^{s+1} - 1$  in  $P_4$ , then either  $\omega(x) = (1^{(s+1)})$  or  $\omega(x) = (3, 2^{(s-1)})$  or  $\omega(x) = (1, 3)$  for  $s = 2$ .*

*Proof.* Obviously, the lemma holds for  $s = 1$ . Suppose  $s \geq 2$ . By a direct computation we see that if  $w$  is a monomial in  $P_4$  such that  $\omega(w) = (1, 3, 2)$  or  $\omega(w) = (1, 1, 3)$ , then  $w$  is strictly inadmissible.

Since  $2^{s+1} - 1$  is odd, we have either  $\omega_1(x) = 1$  or  $\omega_1(x) = 3$ . If  $\omega_1(x) = 1$ , then  $x = x_i y^2$ , where  $y$  is a monomial of degree  $2^s - 1$ . Hence, either  $\omega_1(y) = 1$  or  $\omega_1(y) = 3$ . So, the lemma holds for  $s = 2$ . Suppose that  $s \geq 3$ . If  $\omega_1(y) = 3$ , then  $y = X_i y_1^2$ , where  $y_1$  is a monomial of degree  $2^{s-1} - 2$ . Since  $y_1$  is admissible, using Proposition 2.10, one gets  $\omega_1(y_1) = 2$ . Hence,  $x$  is inadmissible. If  $\omega_1(y) = 1$ , then  $y = x_j y_1^2$ , where  $y_1$  is an admissible monomial of degree  $2^{s-1} - 1$ . By the inductive hypothesis  $\omega(y_1) = (1^{(s-1)})$ . So, we get  $\omega(x) = (1^{(s+1)})$ .

Suppose that  $\omega_1(x) = 3$ . Then,  $x = X_i y^2$ , where  $y$  is an admissible monomial of degree  $2^s - 2$ . Since  $x$  and  $y$  are admissible, by Lemma 5.2.3 and Proposition 2.10,  $\omega(y) = (2^{(s-1)})$ . The lemma is proved.  $\square$

For  $s = 1$ , we have  $(QP_4)_3 = (QP_4^0)_3$ . Hence,  $B_4(3) = \Phi^0(B_3(3))$ . Using Proposition 4.1 and Theorem 4.3, we have

$$|\Phi^0(B_3(3))| = 14, |\Phi^0(B_3(7))| = 26, |\Phi^0(B_3(15))| = 38, \\ |\Phi^0(B_3(2^{s+1} - 1))| = 42, \text{ for } s \geq 4.$$

For  $s = 2$ ,  $B_4(7) = B_4(1^{(3)}) \cup B_4(1, 3) \cup B_4(3, 2)$ . By a direct computation, we have  $B_4(1, 3) = \{x_1 X_1^2\}$ ,  $B_4(3, 2) = \Phi(B_3(7))$ .

Recall that

$$B_3(2^{s+1} - 1) = B_3(1^{(s+1)}) \cup \psi(\Phi(B_2(2^s - 2))),$$

where  $B_2(2^s - 2) = \{x_1^{2^{s-1}-1} x_2^{2^{s-1}-1}\}$ . Hence,  $B_3(3, 2^{(s-1)}) = \psi(\Phi(B_2(2^s - 2)))$ .

**Proposition 5.3.2.** *For any  $s \geq 3$ ,  $B_4(3, 2^{(s-1)}) = \Phi(B_3(3, 2^{(s-1)})) \cup A(s)$ , where  $A(s)$  is determined as follows:*

$$A(3) = \{x_1^3 x_2^4 x_3 x_4^7, x_1^3 x_2^4 x_3^7 x_4, x_1^3 x_2^7 x_3^4 x_4, x_1^7 x_2^3 x_3^4 x_4, x_1^3 x_2^4 x_3^3 x_4^5\}, \\ A(4) = \{x_1^3 x_2^4 x_3^{11} x_4^{13}, x_1^3 x_2^7 x_3^8 x_4^{13}, x_1^7 x_2^3 x_3^8 x_4^{13}, x_1^7 x_2^7 x_3^8 x_4^9, x_1^7 x_2^7 x_3^9 x_4^8\}, \\ A(s) = \{x_1^3 x_2^4 x_3^{2^s-5} x_4^{2^s-3}\}, s \geq 5.$$

Combining Lemma 5.3.1 and Propositions 4.1, 5.3.2, we have

$$B_4(2^{s+1} - 1) = B_4(1^{(s+1)}) \cup \Phi(B_3(3, 2^{(s-1)})) \cup A(s).$$

The following can easily be proved by a direct computation.

**Lemma 5.3.3.** *The following monomials are strictly inadmissible:*

- i)  $x_i^2 x_j, x_i^3 x_j^4, 1 \leq i < j \leq 4$ .
- ii)  $X_2 x_1^2 x_2^2, X_1 x_1^2 x_i^2, i = 2, 3, 4$ .
- iii)  $x_i^3 x_j^{12} x_k x_\ell^{15}, x_i^3 x_j^4 x_k^9 x_\ell^{15}, x_i^3 x_j^5 x_k^8 x_\ell^{15}, i < j < k, \ell \neq i, j, k$ .
- iv)  $x_1^7 x_2^{11} x_3^{12} x_4, x_1^3 x_2^{12} x_3^3 x_4^{13}, X_j x_1^2 x_2^4 x_3^8 x_4^6, x_1^7 x_2^{11} x_3^4 x_4^8 x_j, \\ x_1^3 x_2^3 x_3^{12} x_4^8 x_i^4 x_j, x_1^3 x_2^3 x_3^{24} x_4^{29} x_i^4, i = 1, 2, j = 3, 4$ .

*Proof of Proposition 5.3.2.* By a direct computation using Lemmas 5.3.1, 5.3.3 and Theorem 2.9 we see that if  $x$  is a monomial of degree  $2^{s+1} - 1$  in  $P_4$  and  $x \notin \Phi(B_3(3, 2^{(s-1)})) \cup A(s)$ , then there is a monomial  $w$  which is given in Lemma 5.3.3 such that  $x = wy^{2^u}$  for some monomial  $y$  and integer  $u > 1$ . Hence,  $x$  is inadmissible.



Now we prove that the set  $[\Phi^+(B_3(3, 2^{(s-1)})) \cup A(s)]$  is linearly independent in  $QP_4^+$ . For  $s = 3$ , we have  $|\Phi^+(B_3(3, 2, 2)) \cup A(3)| = 36$ . Suppose there is a linear relation

$$\mathcal{S} = \sum_{1 \leq i \leq 36} \gamma_i d_i \equiv 0, \quad (5.3.3.1)$$

with  $\gamma_i \in \mathbb{F}_2$  and  $d_i = d_{15,i}$ .

By a simple computation, we see that  $B_3(3; 2, 2) = \psi(\Phi(B_2(6)))$  is the set consisting of 6 monomials:

$$v_1 = x_1 x_2^7 x_3^7, v_2 = x_1^3 x_2^5 x_3^7, v_3 = x_1^3 x_2^7 x_3^5, v_4 = x_1^7 x_2 x_3^7, v_5 = x_1^7 x_2^3 x_3^5, v_6 = x_1^7 x_2^7 x_3.$$

By a direct computation, we have

$$\begin{aligned} p_{(1;2)}(\mathcal{S}) &\equiv \gamma_3 v_2 + \gamma_4 v_3 + (\gamma_9 + \gamma_{22}) v_4 + (\gamma_{10} + \gamma_{23}) v_5 + (\gamma_{11} + \gamma_{24}) v_6 \equiv 0, \\ p_{(1;3)}(\mathcal{S}) &\equiv (\gamma_1 + \gamma_{16}) v_1 + \gamma_5 v_2 + (\gamma_7 + \gamma_{20}) v_3 + \gamma_{13} v_5 + (\gamma_{15} + \gamma_{30}) v_6 \equiv 0, \\ p_{(1;4)}(\mathcal{S}) &\equiv (\gamma_2 + \gamma_{19}) v_1 + (\gamma_6 + \gamma_{21} + \gamma_{27}) v_2 + \gamma_8 v_3 + (\gamma_{12} + \gamma_{29}) v_4 + \gamma_{14} v_5 \equiv 0, \\ p_{(2;3)}(\mathcal{S}) &\equiv (\gamma_1 + \gamma_3 + \gamma_5 + \gamma_9) v_1 + (\gamma_{16} + \gamma_{22}) v_2 \\ &\quad + (\gamma_{18} + \gamma_{20} + \gamma_{23} + \gamma_{26}) v_3 + \gamma_{32} v_5 + (\gamma_{34} + \gamma_{36}) v_6 \equiv 0, \\ p_{(2;4)}(\mathcal{S}) &\equiv (\gamma_2 + \gamma_4 + \gamma_8 + \gamma_{11}) v_1 \\ &\quad + (\gamma_{17} + \gamma_{21}) v_2 + (\gamma_{19} + \gamma_{24} v_3 + \gamma_{31} + \gamma_{35}) v_4 + \gamma_{33} v_5 \equiv 0, \\ p_{(3;4)}(\mathcal{S}) &\equiv (\gamma_{12} + \gamma_{13} + \gamma_{14} + \gamma_{15}) v_1 + (\gamma_{25} + \gamma_{26} + \gamma_{27} + \gamma_{28}) v_2 \\ &\quad + (\gamma_{29} + \gamma_{30}) v_3 + (\gamma_{31} + \gamma_{32} + \gamma_{33} + \gamma_{34}) v_4 + (\gamma_{35} + \gamma_{36}) v_5 \equiv 0. \end{aligned}$$

From these equalities, we obtain

$$\begin{cases} \gamma_j = 0, \quad j = 3, 4, 5, 8, 13, 14, 32, 33, \\ \gamma_1 = \gamma_9 = \gamma_{16} = \gamma_{22}, \quad \gamma_2 = \gamma_{11} = \gamma_{19} = \gamma_{24}, \quad \gamma_7 = \gamma_{20}, \\ \gamma_1 = \gamma_9 = \gamma_{16} = \gamma_{22}, \quad \gamma_{10} = \gamma_{23}, \quad \gamma_{17} = \gamma_{21}, \\ \gamma_{12} = \gamma_{15} = \gamma_{29} = \gamma_{30}, \quad \gamma_{31} = \gamma_{34} = \gamma_{35} = \gamma_{36}, \\ \gamma_6 + \gamma_{21} + \gamma_{27} = \gamma_7 + \gamma_{10} + \gamma_{18} + \gamma_{26} = \gamma_{25} + \gamma_{26} + \gamma_{27} + \gamma_{28} = 0. \end{cases} \quad (5.3.3.2)$$

By a direct computation using (5.3.3.2) and Theorem 2.12, we get

$$\begin{aligned} p_{(1;(2,3))}(\mathcal{S}) &\equiv \gamma_{18} w_3 + \gamma_{26} w_5 + \gamma_{28} w_6 \equiv 0, \\ p_{(1;(2,4))}(\mathcal{S}) &\equiv (\gamma_6 + \gamma_{10} + \gamma_{27}) w_2 + \gamma_{25} w_4 + \gamma_{27} w_5 \equiv 0, \\ p_{(1;(3,4))}(\mathcal{S}) &\equiv (\gamma_{17} + \gamma_{18}) w_1 \\ &\quad + (\gamma_6 + \gamma_7 + \gamma_{17} + \gamma_{25} + \gamma_{26} + \gamma_{27}) w_2 + (\gamma_{17} + \gamma_{28}) w_3 \equiv 0. \end{aligned}$$

Combining the above equalities and (5.3.3.2), one gets  $\gamma_j = 0$  for  $j \neq 1, 2, 9, 11, 12, 15, 16, 19, 22, 24, 29, 30, 31$  and  $\gamma_1 = \gamma_9 = \gamma_{16} = \gamma_{22}$ ,  $\gamma_2 = \gamma_{11} = \gamma_{19} = \gamma_{24}$ ,  $\gamma_{12} = \gamma_{15} = \gamma_{29} = \gamma_{30}$ ,  $\gamma_{31} = \gamma_{34} = \gamma_{35} = \gamma_{36}$ . Hence, the relation (5.3.3.1) becomes

$$\gamma_1 \theta_1 + \gamma_2 \theta_2 + \gamma_{12} \theta_3 + \gamma_{31} \theta_4 \equiv 0, \quad (5.3.3.3)$$

where

$$\begin{aligned} \theta_1 &= d_1 + d_9 + d_{16} + d_{22}, \quad \theta_2 = d_2 + d_{11} + d_{19} + d_{24}, \\ \theta_3 &= d_{12} + d_{15} + d_{29} + d_{30}, \quad \theta_4 = d_{31} + d_{34} + d_{35} + d_{36}. \end{aligned}$$

Now, we prove that  $\gamma_1 = \gamma_2 = \gamma_{12} = \gamma_{31} = 0$ .

The proof is divided into 4 steps.

*Step 1.* Under the homomorphism  $\varphi_1$ , the image of (5.3.3.3) is

$$\gamma_1\theta_1 + \gamma_2\theta_2 + \gamma_{12}\theta_3 + \gamma_{31}(\theta_4 + \theta_3) \equiv 0. \quad (5.3.3.4)$$

Combining (5.3.3.3) and (5.3.3.4), we get

$$\gamma_{31}\theta_3 \equiv 0. \quad (5.3.3.5)$$

If the polynomial  $\theta_3$  is hit, then we have

$$\theta_3 = Sq^1(A) + Sq^2(B) + Sq^4(C),$$

for some polynomials  $A \in (P_4^+)_{14}, B \in (P_4^+)_{13}, C \in (P_4^+)_{11}$ . Let  $(Sq^2)^3$  act on the both sides of this equality. We get

$$(Sq^2)^3(\theta_3) = (Sq^2)^3Sq^4(C),$$

By a direct calculation, we see that the monomial  $x = x_1^8x_2^7x_3^4x_4^2$  is a term of  $(Sq^2)^3(\theta_3)$ . If this monomial is a term of  $(Sq^2)^3Sq^4(y)$  for a monomial  $y \in (P_4^+)_{11}$ , then  $y = x_2^7f_2(z)$  with  $z \in P_3$  and  $\deg z = 4$ . Using the Cartan formula, we see that  $x$  is a term of  $x_2^7(Sq^2)^3Sq^4(z) = x_2^7(Sq^2)^3(z^2) = 0$ . Hence,

$$(Sq^2)^3(\theta_3) \neq (Sq^2)^3Sq^4(C),$$

for all  $C \in (P_4^+)_{11}$  and we have a contradiction. So,  $[\theta_3] \neq 0$  and  $\gamma_{31} = 0$ .

*Step 2.* Since  $\gamma_{31} = 0$ , the homomorphism  $\varphi_2$  sends (5.3.3.3) to

$$\gamma_1\theta_1 + \gamma_2\theta_2 + \gamma_{12}\theta_4 \equiv 0. \quad (5.3.3.6)$$

Using the relation (5.3.3.6) and by the same argument as given in Step 1, we get  $\gamma_{12} = 0$ .

*Step 3.* Since  $\gamma_{31} = \gamma_{12} = 0$ , the homomorphism  $\varphi_3$  sends (5.3.3.3) to

$$\gamma_1[\theta_1] + \gamma_2[\theta_3] = 0. \quad (5.3.3.7)$$

Using the relation (5.3.3.7) and by the same argument as given in Step 2, we obtain  $\gamma_3 = 0$ .

*Step 4.* Since  $\gamma_{31} = \gamma_{12} = \gamma_2 = 0$ , the homomorphism  $\varphi_4$  sends (5.3.3.3) to

$$\gamma_1\theta_2 = 0.$$

Using this relation and by the same argument as given in Step 3, we obtain  $\gamma_1 = 0$ .

For  $s \geq 4$ ,  $B_3(3, 2^{(s-1)}) = \psi(\Phi(B_2(2^{s-1} - 2)))$  is the set consisting of 7 monomials:

$$\begin{aligned} v_1 &= x_1x_2^{2^s-1}x_3^{2^s-1}, \quad v_2 = x_1^3x_2^{2^s-3}x_3^{2^s-1}, \quad v_3 = x_1^3x_2^{2^s-1}x_3^{2^s-3}, \quad v_4 = x_1^7x_2^{2^s-5}x_3^{2^s-3}, \\ v_5 &= x_1^{2^s-1}x_2x_3^{2^s-1}, \quad v_6 = x_1^{2^s-1}x_2^3x_3^{2^s-3}, \quad v_7 = x_1^{2^s-1}x_2^{2^s-1}x_3. \end{aligned}$$

Let  $s = 4$ . Then, we have  $|\Phi^+(B_3(3, 2, 2, 2)) \cup A(4)| = 46$ . Suppose there is a linear relation

$$\mathcal{S} = \sum_{1 \leq j \leq 46} \gamma_j d_j = 0, \quad (5.3.3.8)$$

with  $\gamma_j \in \mathbb{F}_2$  and  $d_i = d_{31,i}$ .

By a direct computation using Theorem 2.12, we have

$$\begin{aligned}
p_{(1;2)}(\mathcal{S}) &\equiv \gamma_3 w_2 + \gamma_4 w_3 + (\gamma_9 + \gamma_{25}) w_4 + \gamma_{12} w_5 + \gamma_{13} w_6 + \gamma_{14} w_7 \equiv 0, \\
p_{(1;3)}(\mathcal{S}) &\equiv (\gamma_1 + \gamma_{19}) w_1 + \gamma_5 w_2 + (\gamma_7 + \gamma_{23} + \gamma_{37} + \gamma_{39}) w_3 \\
&\quad + (\gamma_{10} + \gamma_{28}) w_4 + \gamma_{16} w_6 + \gamma_{18} w_7 \equiv 0, \\
p_{(1;4)}(\mathcal{S}) &\equiv (\gamma_2 + \gamma_{22}) w_1 + (\gamma_6 + \gamma_{24} + \gamma_{27} + \gamma_{29} + \gamma_{32} + \gamma_{40}) w_2 \\
&\quad + \gamma_8 w_3 + \gamma_{11} w_4 + (\gamma_{15} + \gamma_{34}) w_5 + \gamma_{17} w_6 \equiv 0.
\end{aligned}$$

From these equalities, we get

$$\begin{cases} \gamma_j = 0, \ j = 3, 4, 5, 8, 11, 12, 13, 14, 16, 17, 18, \\ \gamma_9 = \gamma_{25}, \ \gamma_1 = \gamma_{19}, \ \gamma_7 + \gamma_{23} + \gamma_{37} + \gamma_{39} = 0, \ \gamma_{10} = \gamma_{28}, \\ \gamma_2 = \gamma_{22}, \ \gamma_6 + \gamma_{24} + \gamma_{27} + \gamma_{29} + \gamma_{32} + \gamma_{40} = 0, \ \gamma_{15} + \gamma_{34} = 0. \end{cases} \quad (5.3.3.9)$$

Using the relations (5.3.3.9) and Theorem 2.12, we obtain

$$\begin{aligned}
p_{(2;3)}(\mathcal{S}) &\equiv \gamma_1 w_1 + \gamma_1 w_2 + (\gamma_9 + \gamma_{10} + \gamma_{21} + \gamma_{23} + \gamma_{26} + \gamma_{31} + \gamma_{39}) w_3 \\
&\quad + (\gamma_{35} + \gamma_{37}) w_4 + \gamma_{43} w_6 \equiv 0, \\
p_{(2;4)}(\mathcal{S}) &\equiv \gamma_2 w_1 + \gamma_{45} w_7 + (\gamma_{20} + \gamma_{24} + \gamma_{38} + \gamma_{40}) w_2 + \gamma_2 w_3 + \gamma_{36} w_4 \\
&\quad + (\gamma_{42} + \gamma_{46}) w_5 + \gamma_{44} w_6 \equiv 0, \\
p_{(3;4)}(\mathcal{S}) &\equiv \gamma_{15} w_1 + (\gamma_{30} + \gamma_{31} + \gamma_{32} + \gamma_{33}) w_2 \\
&\quad + \gamma_{15} w_3 + \gamma_{41} w_4 + (\gamma_{42} + \gamma_{43} + \gamma_{44} + \gamma_{45}) w_5 + \gamma_{42} w_6 \equiv 0.
\end{aligned}$$

From these equalities, we get

$$\begin{cases} \gamma_j = 0, \ j = 1, 2, 15, 36, 41, 42, 43, 44, 45, 46, \\ \gamma_{10} + \gamma_{21} + \gamma_{23} + \gamma_{26} + \gamma_{31} + \gamma_{39} = 0, \\ \gamma_{35} = \gamma_{37}, \ \gamma_{20} + \gamma_{24} + \gamma_{38} + \gamma_{40} = 0, \\ \gamma_{30} + \gamma_{31} + \gamma_{32} + \gamma_{33} = 0. \end{cases} \quad (5.3.3.10)$$

By a direct computation using (5.3.3.9), (5.3.3.10) and Theorem 2.12, we have

$$\begin{aligned}
p_{(1;(2,3))}(\mathcal{S}) &\equiv (\gamma_7 + \gamma_{21} + \gamma_{23} + \gamma_{39}) w_3 + \gamma_{26} w_4 + \gamma_{31} w_6 + \gamma_{33} w_7 \equiv 0, \\
p_{(1;(2,4))}(\mathcal{S}) &\equiv (\gamma_6 + \gamma_9 + \gamma_{20} + \gamma_{24} + \gamma_{27} + \gamma_{29} + \gamma_{32} + \gamma_{38} + \gamma_{40}) w_2 \\
&\quad + \gamma_{27} w_4 + \gamma_{30} w_5 + \gamma_{32} w_6 \equiv 0, \\
p_{(1;(3,4))}(\mathcal{S}) &\equiv (\gamma_6 + \gamma_{10} + \gamma_{23} + \gamma_{24} + \gamma_{26} + \gamma_{27} + \gamma_{29} + \gamma_{30} + \gamma_{31} + \gamma_{32}) w_2 \\
&\quad + (\gamma_7 + \gamma_{23} + \gamma_{24} + \gamma_{33} + \gamma_{35} + \gamma_{38} + \gamma_{39} + \gamma_{40}) w_3 \\
&\quad + (\gamma_{20} + \gamma_{21} + \gamma_{35}) w_1 + \gamma_{29} w_4 \equiv 0, \\
p_{(2;(3,4))}(\mathcal{S}) &\equiv (\gamma_{10} + \gamma_{20} + \gamma_{23} + \gamma_{24} + \gamma_{29} + \gamma_{30} + \gamma_{35} + \gamma_{38} + \gamma_{39} + \gamma_{40}) w_2 \\
&\quad + (\gamma_9 + \gamma_{10} + \gamma_{21} + \gamma_{23} + \gamma_{24} + \gamma_{26} + \gamma_{27} + \gamma_{29} + \gamma_{31} + \gamma_{32}) w_3 \\
&\quad + (\gamma_6 + \gamma_7 + \gamma_9 + \gamma_{10}) w_1 + \gamma_{38} w_4 \equiv 0.
\end{aligned}$$

Combining the above equalities, (5.3.3.9) and (5.3.3.10), we get

$$\begin{cases} \gamma_j = 0, \ j \neq 7, 10, 21, 23, 24, 28, 35, 37, 39, 40, \\ \gamma_7 = \gamma_{10} = \gamma_{28}, \ \gamma_{21} = \gamma_{35} = \gamma_{37}, \\ \gamma_7 + \gamma_{21} + \gamma_{23} + \gamma_{39} = 0. \end{cases} \quad (5.3.3.11)$$

Hence, we obtain

$$\gamma_7\theta_1 + \gamma_{21}\theta_2 + \gamma_{39}\theta_3 + \gamma_{24}\theta_4 \equiv 0, \quad (5.3.3.12)$$

where

$$\begin{aligned} \theta_1 &= d_7 + d_{10} + d_{23} + d_{28}, \\ \theta_2 &= d_{21} + d_{23} + d_{35} + d_{37}, \\ \theta_3 &= d_{23} + d_{39}, \quad \theta_4 = d_{24} + d_{40}. \end{aligned}$$

Now, we prove  $\gamma_7 = \gamma_{21} = \gamma_{24} = \gamma_{39} = 0$ . The proof is divided into 4 steps.

*Step 1.* The homomorphism  $\varphi_1$  sends (5.3.3.12) to

$$\gamma_7\theta_1 + \gamma_{21}(\theta_2 + \theta_1) + \gamma_{24}\theta_3 + \gamma_{39}\theta_4 \equiv 0. \quad (5.3.3.13)$$

Combining (5.3.3.12) and (5.3.3.13) gives

$$\gamma_{25}\theta_1 \equiv 0. \quad (5.3.3.14)$$

By an analogous argument as given in the proof of the proposition for the case  $s = 3$ ,  $[\theta_1] \neq 0$ . So, we get  $\gamma_{21} = 0$ .

*Step 2.* Applying the homomorphism  $\varphi_2$  to (5.3.3.8), we obtain

$$\gamma_7\theta_2 + \gamma_{24}\theta_3 + \gamma_{39}\theta_4 = 0. \quad (5.3.3.15)$$

Using (5.3.3.15) and by a same argument as given in Step 1, we get  $\gamma_7 = 0$ .

*Step 3.* Under the homomorphism  $\varphi_3$ , the image of (5.3.3.8) is

$$\gamma_{24}[\theta_2] + \gamma_{39}[\theta_4] = 0. \quad (5.3.3.16)$$

Using (5.3.3.16) and by a same argument as given in Step 3, we obtain  $\gamma_{24} = 0$ .

*Step 4.* Since  $\gamma_7 = \gamma_{22} = \gamma_{24} = 0$ , the homomorphism  $\varphi_3$  sends (5.3.3.8) to

$$\gamma_{39}[\theta_3] = 0.$$

From this equality and by a same argument as given in Step 3, we get  $\gamma_{39} = 0$ .

For  $s \geq 5$ ,  $|\Phi^+(B_3(3, 2^{(s-1)})) \cup A(s)| = 43$ . Suppose that there is a linear relation

$$\mathcal{S} = \sum_{1 \leq j \leq 43} \gamma_j d_j \equiv 0, \quad (5.3.3.17)$$

with  $\gamma_j \in \mathbb{F}_2$ .

Using the relations  $p_{(j;J)}(\mathcal{S}) \equiv 0$ , for  $(j; J) \in \mathcal{N}_4$  and the admissible monomials  $v_i$ ,  $i = 1, 2, \dots, 7$ , we obtain  $\gamma_j = 0$  for any  $j$ . The proposition is proved.  $\square$

#### 5.4. The case of degree $2^{s+t+1} + 2^{s+1} - 3$ .

First of all, we determine the  $\omega$ -vector of an admissible monomial of degree  $n = 2^{s+t+1} + 2^{s+1} - 3$  for any positive integers  $s, t$ .

**Lemma 5.4.1.** *Let  $x$  be a monomial of degree  $2^{s+t+1} + 2^{s+1} - 3$  in  $P_4$  with  $s, t$  positive integers. If  $x$  is admissible, then either  $\omega(x) = (3^{(s)}, 1^{(t+1)})$  or  $\omega(x) = (3^{(s+1)}, 2^{(t-1)})$ .*

*Proof.* Observe that the monomial  $z = x_1^{2^{s+t+1}-1} x_2^{2^s-1} x_3^{2^s-1}$  is the minimal spike of degree  $2^{s+t+1} + 2^{s+1} - 3$  in  $P_4$  and  $\omega(z) = (3^{(s)}, 1^{(t+1)})$ . Since  $x$  is admissible and  $2^{s+t+1} + 2^{s+1} - 3$  is odd, using Theorem 2.12, we obtain  $\omega_1(x) = 3$ . Using Theorem 2.12 and Proposition 2.10, we get  $\omega_i(x) = 3$  for  $i = 1, 2, \dots, s$ .

Let  $x' = \prod_{i \geq 1} X_{i+s-1(x)}^{2^{i-1}}$ . Then,  $\omega_i(x') = \omega_{i+s}(x)$ ,  $i \geq 1$  and  $\deg(x') = 2^{t+1} - 1$ . Since  $x$  is admissible, using Theorem 2.9, we see that  $x'$  is also admissible. By

Lemmas 5.3.1, either  $\omega(x') = (1^{t+1})$  or  $\omega(x') = (3, 2^{t-1})$  or  $\omega(x') = (1, 3)$  for  $t = 2$ . By a direct computation we see that if  $\omega(x') = (1, 3)$ , then  $x$  is inadmissible. So, the lemma is proved.  $\square$

Using Theorem 1.3, we easily obtain the following.

**Proposition 5.4.2.** *For any positive integers  $s, t$  with  $s \geq 3$ ,  $\Phi(B_3(n))$  is a minimal set of generators for  $\mathcal{A}$ -module  $P_4$  in degree  $n = 2^{s+t+1} + 2^{s+1} - 3$ .*

Hence, it suffices to consider the subcases  $s = 1$  and  $s = 2$ .

#### 5.4.1. The subcase $s = 1$ .

For  $s = 1$ ,  $n = 2^{t+2} + 1 = (2^{t+2} - 1) + (2 - 1) + (2 - 1)$ . Hence,  $\mu(2^{t+2} + 1) = 3$  and Kameko's homomorphism

$$\widetilde{Sq}_*^0 : (QP_3)_{2^{t+2}+1} \rightarrow (QP_3)_{2^{t+1}-1}$$

is an isomorphism. So, we get

$$B_3(n) = \psi(B_3(2^{t+1} - 1)) = \psi(B_3(1^{t+1})) \cup \psi(B_3(3, 2^{t-1})).$$

**Proposition 5.4.3.** *For any positive integer  $t$ ,  $C_4(n) = \Phi(B_3(n)) \cup B(t)$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree  $n = 2^{t+2} + 1$ , where the set  $B(t)$  is determined as follows:*

$$\begin{aligned} B(1) &= \{x_1^3 x_2^4 x_3 x_4\}, & B(2) &= \{x_1^3 x_2^5 x_3^8 x_4\}, \\ B(3) &= \{x_1^3 x_2^7 x_3^{11} x_4^{12}, x_1^7 x_2^3 x_3^{11} x_4^{12}, x_1^7 x_2^{11} x_3^3 x_4^{12}, x_1^7 x_2^7 x_3^8 x_4^{11}, x_1^7 x_2^7 x_3^{11} x_4^8\}, \\ B(t) &= \{x_1^3 x_2^7 x_3^{2^{t+1}-5} x_4^{2^{t+1}-4}, x_1^7 x_2^3 x_3^{2^{t+1}-5} x_4^{2^{t+1}-4}, \\ &\quad x_1^7 x_2^{2^{t+1}-5} x_3^3 x_4^{2^{t+1}-4}, x_1^7 x_2^7 x_3^{2^{t+1}-8} x_4^{2^{t+1}-5}\}, \text{ for } t > 3. \end{aligned}$$

The following lemma is proved by a direct computation.

**Lemma 5.4.4.** *The following monomials are strictly inadmissible:*

- i)  $X_2 x_1^2 x_2^{12}, X_3^3 x_3^4 x_i^4, i = 1, 2, X_j x_1^2 x_2^4 x_j^8, X_2^3 x_2^4 x_j^4, j = 3, 4.$
- ii)  $X_3 x_1^2 x_2^4 x_3^{24}, X_3 x_1^2 x_2^4 x_j^8 x_4^{16}, j = 3, 4.$
- iii)  $X_3 X_2^2 x_1^4 x_2^8 x_4^{12}, X_4 X_2^2 x_1^4 x_2^8 x_3^{12}, X_4 X_3^2 x_1^4 x_2^{12} x_3^8, X_4 X_3^2 x_1^4 x_2^4 x_3^8.$
- iv)  $X_j^3 x_i^4 x_j^8 x_m^{12}, 1 \leq i < j \leq 4, m \neq i, j.$
- v)  $X_j X_2^2 x_1^4 x_3^4 x_2^8 x_4^8, j = 3, 4, X_j^3 2x_1^4 x_3^4 x_2^8 x_4^8, j = 2, 4.$
- vi)  $X_3^3 x_1^4 x_2^4 x_3^{24} x_4^{24}, X_3^3 x_1^4 x_2^4 x_i^8 x_3^{16} x_4^{16}, X_4 X_2^2 x_1^4 x_2^4 x_i^8 x_4^{16} x_3^{16}, i = 1, 2,$   
 $X_j^3 x_1^{12} x_2^{12} x_3^{16} x_4^{16}, j = 3, 4, X_4 X_3^2 x_1^{12} x_2^{12} x_3^{16} x_4^{16}.$

*Proof of Proposition 5.4.2.* Let  $x$  be an admissible monomial of degree  $n = 2^{t+2} + 1$ . According to Lemma 5.4.1,  $x = X_i y^2$  with  $y$  a monomial of degree  $2^{t+1} - 1$ . Since  $x$  is admissible, by Theorem 2.12,  $y$  is admissible. By a direct computation, we see that if  $y \in B_4(2^{t+1} - 1)$  and  $X_i y^2 \notin C_4(n)$ , then there is a monomial  $w$  which is given in one of Lemmas 5.1.3, 5.3.3, 5.4.4 such that  $X_i y^2 = w z^{2^u}$  with some positive integer  $u$  and monomial  $z$ . By Theorem 2.9,  $x$  is inadmissible.

Now, we prove the set  $[C_4(n)]$  is linearly independent in  $QP_4$ .

For  $t = 1$ , we have  $|C_4^+(9)| = 18$ . Suppose there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{18} \gamma_i d_i \equiv 0, \quad (5.4.4.1)$$

with  $\gamma_i \in \mathbb{F}_2$ . A direct computation from the relations  $p_{(r;j)}(\mathcal{S}) \equiv 0$ , for  $1 \leq r < j \leq 4$ , we obtain  $\gamma_i = 0$  for  $i \neq 1, 4, 9, 10, 11, 12$  and  $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_{10} = \gamma_{11} = \gamma_{12}$ . Hence, the relation (5.4.4.1) becomes  $\gamma_1 \theta \equiv 0$  where  $\theta = d_1 + d_4 + d_9 + d_{10} + d_{11} + d_{12}$ .

We prove  $\gamma_1 = 0$ . Suppose  $\theta$  is hit. Then we get

$$\theta = Sq^1(A) + Sq^2(B) + Sq^4(C),$$

for some polynomials  $A \in (P_4^+)_8, B \in (P_4^+)_7, C \in (P_4^+)_5$ . Let  $(Sq^2)^3$  act on the both sides of this equality. It is easy to check that  $(Sq^2)^3 Sq^4(C) = 0$  for all  $C \in (P_4^+)_5$ . Since  $(Sq^2)^3$  annihilates  $Sq^1$  and  $Sq^2$ , the right hand side is sent to zero. On the other hand, a direct computation shows

$$(Sq^2)^3(\theta) = (1, 2, 4, 8) + \text{symmetries} \neq 0.$$

Hence, we have a contradiction. So, we obtain  $\gamma_1 = 0$ .

For  $t = 2$ ,  $|C_4^+(17)| = 47$ . Suppose there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{47} \gamma_i d_i \equiv 0, \quad (5.4.4.2)$$

with  $\gamma_i \in \mathbb{F}_2$  and  $d_i = d_{17,i}$ . A direct computation from the relations  $p_{(j;J)}(\mathcal{S}) \equiv 0$ , for  $(j;J) \in \mathcal{N}_4$ , we obtain  $\gamma_i = 0$  for  $i \neq 1, 4, 8, 9, 10, 11, 17, 18$  and  $\gamma_1 = \gamma_2 = \gamma_8 = \gamma_9 = \gamma_{10} = \gamma_{11} = \gamma_{17} = \gamma_{18}$ . Hence, the relation (5.4.4.2) becomes  $\gamma_1 \theta \equiv 0$  where  $\theta = d_1 + d_4 + d_8 + d_{11} + d_{13} + d_{16} + d_{17} + d_{18}$ .

By a same argument as given in the proof of the proposition for  $t = 1$ , we see that  $[\theta] \neq 0$ . Hence,  $\gamma_1 = 0$ .

We have  $|C_4^+(33)| = 84$  for  $t = 3$ , and  $|C_4^+(2^{t+2} + 1)| = 94$  for  $t > 3$ . Suppose there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{84} \gamma_i d_i \equiv 0, \quad (5.4.4.3)$$

with  $\gamma_i \in \mathbb{F}_2$  and  $d_i = d_{33,i}$ . A direct computation from the relations  $p_{(j;J)}(\mathcal{S}) \equiv 0$ , for  $(j;J) \in \mathcal{N}_4$ , we obtain  $\gamma_i = 0$  for all  $i \notin E$  with  $E = \{1, 3, 8, 9, 13, 14, 17, 24, 25, 42, 43, 59, 60, 65, 66, 67\}$  and  $\gamma_i = \gamma_1$  for all  $i \in E$ . Hence, the relation (5.4.4.3) become  $\gamma_1 \theta \equiv 0$  with  $\theta = \sum_{i \in E} d_i$ .

By a same argument as given in the proof of the proposition for  $t = 1$ , we see that  $[\theta] \neq 0$ . Therefore  $\gamma_1 = 0$ .

Now, we prove the set  $[C_4^+(n)]$  is linearly independent for  $t > 3$ . Suppose there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{94} \gamma_i d_i \equiv 0, \quad (5.4.4.4)$$

with  $\gamma_i \in \mathbb{F}_2$  and  $d_i = d_{n,i}$ . By a direct computation from the relations  $p_{(j;J)}(\mathcal{S}) \equiv 0$ , for  $(j;J) \in \mathcal{N}_4$ , we obtain  $\gamma_i = 0$  for all  $i$ .  $\square$

#### 5.4.2. The subcase $s = 2$ .

For  $s = 2$ , we have  $n = 2^{t+3} + 5$ . According to Theorem 1.2, the iterated Kameko homomorphism

$$(\widetilde{Sq}_*^0)^2 : (QP_3)_{2^{t+3}+5} \rightarrow (QP_3)_{2^{t+1}-1}$$

is an isomorphism. So, we get

$$B_3(n) = \psi^2(B_3(2^{t+1} - 1)) = \psi^2(B_3(1^{(t+1)})) \cup \psi^2(\Phi(B_3(3, 2^{(t-1)}))).$$

**Proposition 5.4.5.**

i)  $C_4(21) = \Phi(B_3(21)) \cup \{x_1^7 x_2^9 x_3^2 x_4^3, x_1^7 x_2^9 x_3^3 x_4^2, x_1^3 x_2^7 x_3^8 x_4^3, x_1^7 x_2^3 x_3^8 x_4^3\}$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree 21.

ii) For any integer  $t > 1$ ,  $C_4(n) = \Phi(B_3(n))$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree  $n = 2^{t+3} + 5$ .

The following lemma is proved by a direct computation.

**Lemma 5.4.6.** *The following monomials are strictly inadmissible:*

- i)  $X_2^3 x_3^4, X_i^4 X_j^3, 1 \leq i < j \leq 4, X_2^3 x_1^4 x_2^8$ .
- ii)  $X_3^3 x_i^4 x_3^8 x_4^{16}, X_3^3 x_i^4 x_3^8 x_4^{16}, X_4^3 x_i^4 x_3^8 x_4^{16}, X_4^7 x_i^8 x_4^8, i = 1, 2$ .
- iii)  $x_1^7 x_2^{11} x_3^{17} x_4^2, X_j^3 x_2^8 x_j^{16}, X_j^7 x_3^8 x_4^8, j = 3, 4$ .
- iv)  $x_1^{15} x_2^{15} x_3^{16} x_4^{23}, x_1^{15} x_2^{15} x_3^{23} x_4^{16}, x_1^{15} x_2^{15} x_3^{17} x_4^{22}$ .

*Proof of Proposition 5.4.5.* Let  $x$  be an admissible monomial of degree  $n = 2^{t+3} + 5$ . According to Lemma 5.4.1,  $x = X_i y^2$  with  $y$  a monomial of degree  $2^{t+2} + 1$ . Since  $x$  is admissible, by Theorem 2.12,  $y$  is admissible.

By a direct computation, we see that if  $y \in B_4(2^{t+2} + 1)$  and  $X_i y^2$  does not belong to  $C_4(n)$ , then there is a monomial  $w$  which is given in one of Lemmas 5.1.3, 5.3.3, 5.4.6 such that  $X_i y^2 = w z^{2^u}$  with some positive integer  $u$  and monomial  $z$ . By Theorem 2.9,  $x$  is inadmissible. Hence,  $QP_4(n)$  is generated by the set given in the proposition.

Now, we prove the set  $[C_4^+(n)]$  is linearly independent in  $QP_4$ .

For  $t = 1$ , we have  $|C_4^+(21)| = 66$ . Suppose there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{66} \gamma_i d_i \equiv 0, \quad (5.4.6.1)$$

with  $\gamma_i \in \mathbb{F}_2$  and  $d_i = d_{21,i}$ .

By a simple computation, we see that  $B_3(21)$  is the set consisting of 7 monomials:

$$\begin{aligned} v_1 &= x_1^3 x_2^3 x_3^{15}, \quad v_2 = x_1^3 x_2^7 x_3^{11}, \quad v_3 = x_1^3 x_2^{15} x_3^3, \quad v_4 = x_1^7 x_2^3 x_3^{11}, \\ v_5 &= x_1^7 x_2^{11} x_3^3, \quad v_6 = x_1^{15} x_2^3 x_3^3, \quad v_7 = x_1^7 x_2^7 x_3^7. \end{aligned}$$

A direct computation shows

$$\begin{aligned} p_{(1;2)}(\mathcal{S}) &\equiv \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3 + \gamma_{10} v_4 + \gamma_{11} v_5 + \gamma_{16} v_6 + \gamma_{57} v_7 \equiv 0, \\ p_{(1;3)}(\mathcal{S}) &\equiv \gamma_4 v_1 + \gamma_6 + \gamma_{27} v_2 + (\gamma_8 + \gamma_{30} + \gamma_{49}) v_3 + \gamma_{12} v_4 \\ &\quad + (\gamma_{14} + \gamma_{38} v_5 + \gamma_{17}) v_6 + \gamma_{58} v_7 \equiv 0, \\ p_{(1;4)}(\mathcal{S}) &\equiv (\gamma_5 + \gamma_{26} + \gamma_{48}) v_1 + (\gamma_7 + \gamma_{29} v_2 + \gamma_9) v_3 + (\gamma_{13} + \gamma_{37}) v_4 \\ &\quad + \gamma_{15} v_5 + \gamma_{18} v_6 + \gamma_{59} v_7 \equiv 0, \\ p_{(2;3)}(\mathcal{S}) &\equiv \gamma_{19} v_1 + (\gamma_{21} + \gamma_{27} + \gamma_{32} + \gamma_{60}) v_2 + (\gamma_{23} + \gamma_{30} + \gamma_{34} + \gamma_{38} + \gamma_{40}) v_3 \\ &\quad + \gamma_{43} v_4 + (\gamma_{45} + \gamma_{49} + \gamma_{51}) v_5 + \gamma_{54} v_6 + \gamma_{63} v_7 \equiv 0, \\ p_{(2;4)}(\mathcal{S}) &\equiv (\gamma_{20} + \gamma_{26} + \gamma_{33} + \gamma_{37} + \gamma_{41}) v_1 + \gamma_{22} + \gamma_{29} + \gamma_{35} + \gamma_{61}) v_2 \\ &\quad + \gamma_{24} v_3 + (\gamma_{44} + \gamma_{48} + \gamma_{52}) v_4 + \gamma_{46} v_5 + \gamma_{55} v_6 + \gamma_{64} v_7 \equiv 0, \\ p_{(3;4)}(\mathcal{S}) &\equiv (\gamma_{25} + \gamma_{26} + \gamma_{27} + \gamma_{28} + \gamma_{29} + \gamma_{30} + \gamma_{31}) v_1 \\ &\quad + (\gamma_{36} + \gamma_{37} + \gamma_{38} + \gamma_{39} + \gamma_{62}) v_2 + \gamma_{42} v_3 \\ &\quad + (\gamma_{47} + \gamma_{48} + \gamma_{49} + \gamma_{50} + \gamma_{65}) v_4 + \gamma_{53} v_5 + \gamma_{56} v_6 + \gamma_{66} v_7 \equiv 0. \end{aligned}$$

From the above equalities, we get  $\gamma_i = 0$ , for  $i = 1, 2, 3, 4, 9, 10, 11, 12, 15, 16, 17, 18, 19, 24, 42, 43, 46, 53, 54, 55, 56, 57, 58, 59, 63, 64, 66$  and  $\gamma_6 = \gamma_{27} \cdot \gamma_8 + \gamma_{30} + \gamma_{49} = 0$ ,  $\gamma_{14} = \gamma_{38}$ ,  $\gamma_5 + \gamma_{26} + \gamma_{48} = 0$ ,  $\gamma_7 = \gamma_{29}$ ,  $\gamma_{13} = \gamma_{37}$ ,  $\gamma_6 + \gamma_{21} + \gamma_{32} + \gamma_{60} = 0$ ,  $\gamma_{14} + \gamma_{23} + \gamma_{30} + \gamma_{34} + \gamma_{40}\gamma_{45} + \gamma_{49} + \gamma_{51} = 0$ ,  $\gamma_{20} + \gamma_{26} + \gamma_{33} + \gamma_{37} + \gamma_{41} = 0$ ,  $\gamma_7 + \gamma_{22} + \gamma_{35} + \gamma_{61} = 0$ ,  $\gamma_{44} + \gamma_{48} + \gamma_{52} = 0$ ,  $\gamma_6 + \gamma_7 + \gamma_{25} + \gamma_{26} + \gamma_{28} + \gamma_{30} + \gamma_{31} = 0$ ,  $\gamma_{14} + \gamma_{36} + \gamma_{37} + \gamma_{39} + \gamma_{62} = 0$ ,  $\gamma_{47} + \gamma_{48} + \gamma_{49} + \gamma_{50} + \gamma_{65} = 0$ .

With the aid of the above equalities, we have

$$\begin{aligned} p_{(1;(2,3))}(\mathcal{S}) &\equiv \gamma_{21}v_2 + (\gamma_8 + \gamma_{23} + \gamma_{30} + \gamma_{45} + \gamma_{49})v_3 + \gamma_{32}v_4 \\ &\quad + (\gamma_{34} + \gamma_{45} + \gamma_{49} + \gamma_{51})v_5 + (\gamma_{40} + \gamma_{51})v_6 + \gamma_{60}v_7 \equiv 0, \\ p_{(1;(2,4))}(\mathcal{S}) &\equiv (\gamma_5 + \gamma_{20} + \gamma_{26} + \gamma_{44} + \gamma_{48})v_1 + \gamma_{22}v_2 \\ &\quad + (\gamma_{33} + \gamma_{44} + \gamma_{48} + \gamma_{52})v_4 + \gamma_{35}v_5 + (\gamma_{41} + \gamma_{52})v_6 + \gamma_{61}v_7 \equiv 0. \end{aligned}$$

From this, we obtain  $\gamma_i = 0$ , for  $i = 21, 22, 32, 35, 60, 61$  and  $\gamma_8 + \gamma_{23} + \gamma_{30} + \gamma_{45} + \gamma_{49} = 0$ ,  $\gamma_{34} + \gamma_{45} + \gamma_{49} + \gamma_{51} = 0$ ,  $\gamma_{40} = \gamma_{51}$ ,  $\gamma_5 + \gamma_{20} + \gamma_{26} + \gamma_{44} + \gamma_{48} = 0$ ,  $\gamma_{33} + \gamma_{44} + \gamma_{48} + \gamma_{52} = 0$ ,  $\gamma_{41} = \gamma_{52}$ . By a direct computation using the above equalities, one gets

$$\begin{aligned} p_{(1;(3,4))}(\mathcal{S}) &\equiv (\gamma_5 + \gamma_{25} + \gamma_{26} + \gamma_{47} + \gamma_{48})v_1 + (\gamma_{28} + \gamma_{47} + \gamma_{48} + \gamma_{49} + \gamma_{50})v_2 \\ &\quad + (\gamma_8 + \gamma_{30} + \gamma_{31} + \gamma_{49} + \gamma_{50})v_3 + \gamma_{36}v_4 + \gamma_{39}v_5 + \gamma_{62}v_7 \equiv 0, \\ p_{(2;(3,4))}(\mathcal{S}) &\equiv (\gamma_{13} + \gamma_{20} + \gamma_{25} + \gamma_{26} + \gamma_{33} + \gamma_{36} + \gamma_{40} + \gamma_{41})v_1 + (\gamma_6 + \gamma_7 \\ &\quad + \gamma_{13} + \gamma_{14} + \gamma_{28} + \gamma_{33} + \gamma_{34} + \gamma_{36} + \gamma_{39})v_2 + (\gamma_{14} + \gamma_{23} + \gamma_{30} + \gamma_{31} \\ &\quad + \gamma_{34} + \gamma_{39} + \gamma_{40} + \gamma_{41})v_3 + (\gamma_{44} + \gamma_{47} + \gamma_{48} + \gamma_{51} + \gamma_{52})v_4 \\ &\quad + (\gamma_{45} + \gamma_{49} + \gamma_{50} + \gamma_{51} + \gamma_{52})v_5 + \gamma_{65}v_7 \equiv 0. \end{aligned}$$

So, we obtain  $\gamma_{36} = \gamma_{39} = \gamma_{62} = \gamma_{65} = 0$ ,  $\gamma_5 + \gamma_{25} + \gamma_{26} + \gamma_{47} + \gamma_{48} = 0$ ,  $\gamma_{28} + \gamma_{47} + \gamma_{48} + \gamma_{49} + \gamma_{50} = 0$ ,  $\gamma_8 + \gamma_{30} + \gamma_{31} + \gamma_{49} + \gamma_{50} = 0$ ,  $\gamma_{13} + \gamma_{20} + \gamma_{25} + \gamma_{26} + \gamma_{33} + \gamma_{40} + \gamma_{41} = 0$ ,  $\gamma_6 + \gamma_7 + \gamma_{13} + \gamma_{14} + \gamma_{28} + \gamma_{33} + \gamma_{34} = 0$ ,  $\gamma_{14} + \gamma_{23} + \gamma_{30} + \gamma_{31} + \gamma_{34} + \gamma_{40} + \gamma_{41} = 0$ ,  $\gamma_{44} + \gamma_{47} + \gamma_{48} + \gamma_{51} + \gamma_{52} = 0$ ,  $\gamma_{45} + \gamma_{49} + \gamma_{50} + \gamma_{51} + \gamma_{52} = 0$ .

Combining the above equalities, one gets  $\gamma_i = 0$  for  $i \neq 5, 8, 13, 14, 20, 23, 25, 26, 30, 31, 37, 38, 40, 41, 44, 45, 47, 48, 49, 50, 51$ ,  $\gamma_i = \gamma_5$  for  $i = 8, 13, 14, 37, 38$ ,  $\gamma_i = \gamma_{20}$  for  $i = 23, 44, 45$ ,  $\gamma_i = \gamma_{25}$  for  $i = 40, 47, 51$ ,  $\gamma_i = \gamma_{31}$  for  $i = 41, 50, 52$ ,  $\gamma_{20} + \gamma_{25} + \gamma_{49} = 0$ ,  $\gamma_5 + \gamma_{20} + \gamma_{26} + \gamma_{31} = 0$ ,  $\gamma_{20} + \gamma_{31} + \gamma_{48} = 0$ ,  $\gamma_5 + \gamma_{20} + \gamma_{25} + \gamma_{30} = 0$ .

Substituting the above equalities into the relation (5.4.6.1), we have

$$\gamma_{25}[\theta_1] + \gamma_{31}[\theta_2] + \gamma_5[\theta_3] + \gamma_{20}[\theta_4] = 0, \quad (5.4.6.2)$$

where

$$\begin{aligned} \theta_1 &= d_{25} + d_{30} + d_{40} + d_{47} + d_{49} + d_{51}, \\ \theta_2 &= d_{26} + d_{31} + d_{41} + d_{48} + d_{50} + d_{52}, \\ \theta_3 &= d_5 + d_8 + d_{13} + d_{14} + d_{26} + d_{30} + d_{37} + d_{38}, \\ \theta_4 &= d_{20} + d_{23} + d_{26} + d_{30} + d_{44} + d_{45} + d_{48} + d_{49}. \end{aligned}$$

We need to show that  $\gamma_5 = \gamma_{20} = \gamma_{25} = \gamma_{31} = 0$ . The proof is divided into 4 steps.

*Step 1.* The homomorphism  $\varphi_1$  sends (5.4.6.2) to

$$\gamma_{25}[\theta_1] + \gamma_{31}[\theta_2] + (\gamma_5 + \gamma_{20})[\theta_3] + \gamma_{20}[\theta_4] = 0. \quad (5.4.6.3)$$



Combining (5.4.6.2) and (5.4.6.3) gives

$$\gamma_{20}[\theta_3] = 0.$$

We prove  $[\theta_3] \neq 0$ . We have  $\varphi_2\varphi_3([\theta_1]) = [\theta_3]$ . So, we need only to prove that  $[\theta_1] \neq 0$ . Suppose  $[\theta_1] = 0$ . Then the polynomial  $\theta_1$  is hit and we have

$$\theta_1 = Sq^1(A) + Sq^2(B) + Sq^4(C) + Sq^8(D),$$

for some polynomials  $A \in (P_4^+)_{20}, B \in (P_4^+)_{19}, C \in (P_4^+)_{17}, D \in (P_4^+)_{13}$ .

Let  $(Sq^2)^3$  act on the both sides of this equality. Since  $(Sq^2)^3Sq^1 = 0$  and  $(Sq^2)^3Sq^2 = 0$ , we get

$$(Sq^2)^3(\theta_3) = (Sq^2)^3Sq^4(C) + (Sq^2)^3Sq^8(D).$$

By a direct computation, we see that the monomial  $x = x_1^7x_2^{12}x_3^2x_4^6$  is a term of  $(Sq^2)^3(\theta_1)$ . If this monomial is a term of  $(Sq^2)^3Sq^8(y)$ , then  $y = x_1^7f_1(z)$  with  $z$  a monomial of degree 6 in  $P_3$  and  $x$  is a term of  $x_1^7(Sq^2)^3Sq^8(f_1(z)) = 0$ . So, the monomial  $x$  is not a term of  $(Sq^2)^3Sq^8(D)$  for all  $D \in (P_4^+)_{13}$ .

If this monomial is a term of  $(Sq^2)^3Sq^4(y)$ , where the monomial  $y$  is a term of  $C$ , then  $y = x_1^7f_1(z)$  with  $z$  a monomial of degree 10 in  $P_3$  and  $x$  is a term of  $x_1^7(Sq^2)^3Sq^4(f_1(z)) = 0$ . By a direct computation, we see that either  $x_1^7x_2^6x_3x_4^3$  or  $x_1^7x_2^5x_3^2x_4^3$  is a term of  $C$ .

If  $x_1^7x_2^6x_3x_4^3$  is a term of  $C$  then

$$(Sq^2)^3(\theta_1 + Sq^4(x_1^7x_2^6x_3x_4^3)) = (Sq^2)^3(Sq^4(C') + Sq^8(D)),$$

where  $C' = C + x_1^7x_2^6x_3x_4^3$ . The monomial  $x' = x_1^{16}x_2^6x_3^2x_4^3$  is a term of the polynomial  $(Sq^2)^3(\theta_1 + Sq^4(x_1^7x_2^6x_3x_4^3))$ . If  $x'$  is a term of the polynomial  $(Sq^2)^3Sq^8(y')$ , with  $y'$  a monomial in  $(P_4^+)_{13}$ . Then  $y' = x_1^ax_2^bx_3^cx_4^3$  with  $a \geq 7, b \geq 3, c > 0$ . This contradicts with the fact that  $\deg y' = 13$ . So,  $x'$  is not a term of  $(Sq^2)^3Sq^8(D)$  for all  $D \in (P_4^+)_{13}$ . Hence,  $x'$  is a term of  $(Sq^2)^3(Sq^4(C'))$ . By a direct computation, we see that either  $x_1^7x_2^6x_3x_4^3$  or  $x_1^7x_2^5x_3^2x_4^3$  is a term of  $C'$ . Since  $x_1^7x_2^6x_3x_4^3$  is not a term of  $C'$ , the monomial  $x_1^7x_2^5x_3^2x_4^3$  is a term of  $C'$ . Then we have

$$(Sq^2)^3(\theta_1 + Sq^4(x_1^7x_2^6x_3x_4^3 + x_1^7x_2^5x_3^2x_4^3)) = (Sq^2)^3(Sq^4(C'') + Sq^8(D)),$$

where  $C'' = C' + x_1^7x_2^5x_3^2x_4^3 = C + x_1^7x_2^6x_3x_4^3 + x_1^7x_2^5x_3^2x_4^3$ . Now the monomial  $x = x_1^7x_2^{12}x_3^2x_4^6$  is a term of

$$(Sq^2)^3(\theta_1 + Sq^4(x_1^7x_2^6x_3x_4^3 + x_1^7x_2^5x_3^2x_4^3)).$$

Hence, either  $x_1^7x_2^6x_3x_4^3$  or  $x_1^7x_2^5x_3^2x_4^3$  is a term of  $C''$ . On the other hand, the two monomials  $x_1^7x_2^6x_3x_4^3$  and  $x_1^7x_2^5x_3^2x_4^3$  are not the terms of  $C''$ . We have a contradiction. So, one gets  $\gamma_{20} = 0$ .

*Step 2.* Since  $\gamma_{20} = 0$ , the homomorphism  $\varphi_2$  sends (5.4.6.3) to

$$\gamma_{25}[\theta_1] + \gamma_{31}[\theta_2] + \gamma_5[\theta_3] = 0. \quad (5.4.6.4)$$

Using (5.4.6.4) and the result in Step 1, we get  $\gamma_5 = 0$ .

*Step 3.* The homomorphism  $\varphi_3$  sends (5.4.6.3) to

$$\gamma_{25}[\theta_4] + \gamma_{31}[\theta_2] = 0. \quad (5.4.6.5)$$

Using the relation (5.4.6.5) and the result in Step 2, we obtain  $\gamma_{25} = 0$ .

*Step 4.* Since  $\varphi_4([\theta_2]) = [\theta_1]$ , we have

$$\gamma_{31}[\theta_1] = 0.$$

Using this equality and by a same argument as given in Step 3, we get  $\gamma_{31} = 0$ .

For  $t > 1$ , we have  $|C_4^+(n)| = m(t)$  with  $m(2) = 95, m(3) = 128$  and  $m(t) = 139$  for  $t \geq 4$ . Suppose there is a linear relation

$$S = \sum_{i=1}^{m(t)} \gamma_i d_i \equiv 0, \quad (5.4.6.6)$$

with  $\gamma_i \in \mathbb{F}_2$  and  $d_i = d_{n,i}$ . A direct computation from the relations  $p_{(j;J)}(S) \equiv 0$ , for  $(j;J) \in \mathcal{N}_4$ , we obtain  $\gamma_i = 0$  for all  $i$ . The proposition is proved.  $\square$

### 5.5. The case of degree $n = 2^{s+t} + 2^s - 2$ .

For  $s \geq 1$  and  $t \geq 2$ , the space  $(QP_4)_n$  has been determined in [33]. Hence, in this subsection we need only to compute  $(QP_4)_n$  for  $n = 2^{s+1} + 2^s - 2$  with  $s > 1$ .

Recall that, the homomorphism

$$\widetilde{Sq}_*^0 : (QP_4)_{2^{s+1}+2^s-2} \rightarrow (QP_4)_{2^s+2^{s-1}-3}$$

is an epimorphism. Hence, we have

$$(QP_4)_{2m+4} \cong (QP_4)_m \oplus (QP_4^0)_{2m+4} \oplus (\text{Ker} \widetilde{Sq}_*^0 \cap (QP_4^+)_{2m+4}),$$

where  $m = 2^s + 2^{s-1} - 3$ . So, it suffices to compute  $\text{Ker} \widetilde{Sq}_*^0 \cap (QP_4^+)_n$  for  $s > 1$ .

For  $s > 1$ , denote by  $C(s)$  the set of all the following monomials:

$$\begin{aligned} & x_1 x_2 x_3^{2^s-2} x_4^{2^{s+1}-2}, \quad x_1 x_2 x_3^{2^{s+1}-2} x_4^{2^s-2}, \quad x_1 x_2^{2^s-2} x_3 x_4^{2^{s+1}-2}, \\ & x_1 x_2^{2^{s+1}-2} x_3 x_4^{2^s-2}, \quad x_1 x_2^2 x_3^{2^s-4} x_4^{2^{s+1}-1}, \quad x_1 x_2^2 x_3^{2^{s+1}-1} x_4^{2^s-4}, \\ & x_1 x_2^{2^{s+1}-1} x_3^2 x_4^{2^s-4}, \quad x_1^{2^{s+1}-1} x_2 x_3^2 x_4^{2^s-4}, \quad x_1 x_2^2 x_3^{2^{s+1}-3} x_4^{2^s-2}, \\ & x_1 x_2^3 x_3^{2^{s+1}-4} x_4^{2^s-2}, \quad x_1^3 x_2 x_3^{2^{s+1}-4} x_4^{2^s-2}. \end{aligned}$$

For  $s > 2$ , denote by  $D(s)$  the set of all the following monomials:

$$\begin{aligned} & x_1 x_2^2 x_3^{2^s-3} x_4^{2^{s+1}-2}, \quad x_1 x_2^2 x_3^{2^s-1} x_4^{2^{s+1}-4}, \quad x_1 x_2^2 x_3^{2^{s+1}-4} x_4^{2^s-1}, \\ & x_1 x_2^{2^s-1} x_3^2 x_4^{2^{s+1}-4}, \quad x_1^{2^s-1} x_2 x_3^2 x_4^{2^{s+1}-4}, \quad x_1 x_2^3 x_3^{2^s-4} x_4^{2^{s+1}-2}, \\ & x_1 x_2^3 x_3^{2^{s+1}-2} x_4^{2^s-4}, \quad x_1^3 x_2 x_3^{2^s-4} x_4^{2^{s+1}-2}, \quad x_1^3 x_2 x_3^{2^{s+1}-2} x_4^{2^s-4}, \\ & x_1 x_2^3 x_3^{2^s-2} x_4^{2^{s+1}-4}, \quad x_1^3 x_2 x_3^{2^s-2} x_4^{2^{s+1}-4}, \quad x_1 x_2^{2^{s+1}-3} x_3^2 x_4^{2^s-4}, \\ & x_1^3 x_2^{2^s-3} x_3^2 x_4^{2^{s+1}-4}, \quad x_1^3 x_2^5 x_3^{2^{s+1}-6} x_4^{2^s-4}. \end{aligned}$$

Set  $E(2) = C(2) \cup \{x_1^3 x_2^4 x_3 x_4\}$ ,  $E(3) = C(3) \cup D(3) \cup \{x_1^3 x_2^5 x_3^6 x_4^8\}$  and  $E(s) = C(s) \cup D(s) \cup \{x_1^3 x_2^5 x_3^{2^s-6} x_4^{2^{s+1}-4}\}$ , for  $s > 3$ .

**Proposition 5.5.1.** *For any integer  $s > 1$ ,  $E(s) \cup \Phi^0(B_3(n)) \cup \psi(B_4(m))$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree  $n = 2m + 4$  with  $m = 2^s + 2^{s-1} - 3$ .*

**Lemma 5.5.2.** *Let  $s$  be a positive integer and let  $x$  be an admissible monomial of degree  $n = 2^{s+1} + 2^s - 2$  in  $P_4$ . If  $[x] \in \text{Ker} \widetilde{Sq}_*^0$ , then  $\omega(x) = (2^s, 1)$ .*

*Proof.* We prove the lemma by induction on  $s$ . Since  $n = 2^{s+1} + 2^s - 2$  is even, we get either  $\omega_1(x) = 0$  or  $\omega_1(x) = 2$  or  $\omega_1(x) = 4$ . If  $\omega_1(x) = 0$ , then  $x = Sq^1(y)$  for some monomial  $y$ . If  $\omega_1(x) = 4$ , then  $x = X_0 y^2$  for some monomial  $y$ . Since  $x$  is admissible,  $y$  also is admissible. This implies  $\text{Ker} \widetilde{Sq}_*^0([x]) = [y] \neq 0$  and we have a contradiction. So,  $\omega_1(x) = 2$  and  $x = x_i x_j y^2$  with  $1 \leq i < j \leq 4$ , and  $y$  a monomial of degree  $2^s + 2^{s-1} - 2$  in  $P_4$ . Since  $x$  is admissible,  $y$  is also admissible.

If  $s = 1$ , then  $\deg y = 1$ . Hence, the lemma holds. Now, the lemma follows from Proposition 2.10 and the inductive hypothesis.  $\square$

The following is proved by a direct computation.

**Lemma 5.5.3.** *The following monomials are strictly inadmissible:*

- i)  $x_i^2 x_j x_m, x_i^3 x_j^4 x_m^3, x_i^7 x_j^7 x_m^8, 1 \leq i < j < m \leq 4$ .
- ii)  $x_1 x_2^7 x_3^{10} x_4^4, x_1^7 x_2 x_3^{10} x_4^4, x_1 x_2^6 x_3^7 x_4^8, x_1 x_2^7 x_3^6 x_4^8, x_1^7 x_2 x_3^6 x_4^8, x_1^3 x_2^3 x_3^4 x_4^{12}, x_1^3 x_2^3 x_3^{12} x_4^4, x_1^7 x_2^9 x_3^2 x_4^4, x_1^7 x_2^8 x_3^3 x_4^4, x_1^3 x_2^5 x_3^8 x_4^6$ .

*Proof of Proposition 5.5.1.* Let  $x$  be an admissible monomial of degree  $n = 2^{s+1} + 2^s - 2$  in  $P_4$  and  $[x] \in \widetilde{\text{Ker}} S q_*^0$ . By Lemma 5.5.2,  $\omega_i(x) = 2$ , for  $1 \leq i \leq s$ ,  $\omega_{s+1}(x) = 1$  and  $\omega_i(x) = 0$  for  $i > s + 1$ . By induction on  $s$ , we see that if  $x \notin E(s) \cup \Phi^0(B_3(n))$  then there is a monomial  $w$  which is given in one of Lemmas 5.2.3, 5.5.3 such that  $x = wy^{2^u}$  for some monomial  $y$  and positive integer  $u$ . By Theorem 2.9,  $x$  is inadmissible. Hence,  $\widetilde{\text{Ker}} S q_*^0$  is spanned by the set  $[E(s) \cup \Phi^0(B_3(n))]$ . Now, we prove that set  $[E(s) \cup \Phi^0(B_3(n))]$  is linearly independent.

It suffices to prove that the set  $[E(s)]$  is linearly independent. For  $s = 2$ ,  $|E(2)| = 12$ . Suppose there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{12} \gamma_i d_i \equiv 0, \quad (5.5.3.1)$$

with  $\gamma_i \in \mathbb{F}_2$  and  $d_i = d_{10,i}$ . By a direct computation from the relations  $p_{(1;j)}(\mathcal{S}) \equiv 0$ , for  $j = 1, 2, 3$ , we obtain  $\gamma_i = 0$  for all  $i$ .

For  $s > 2$ ,  $|E(s)| = 26$ . Suppose there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{26} \gamma_i d_i \equiv 0, \quad (5.5.3.2)$$

with  $\gamma_i \in \mathbb{F}_2$  and  $d_i = d_{n,i}$ . By a direct computation from the relations  $p_{(r;j)}(\mathcal{S}) \equiv 0$ , for  $1 \leq r < j \leq 4$ , we obtain  $\gamma_i = 0$  for all  $i$ . The proposition is proved.  $\square$

## 5.6. The case of degree $2^{s+t+u} + 2^{s+t} + 2^s - 3$ .

First, we determine the  $\omega$ -vector of an admissible monomial of degree  $n = 2^{s+t+u} + 2^{s+t} + 2^s - 3$ .

**Lemma 5.6.1.** *If  $x$  is an admissible monomial of degree  $2^{s+t+u} + 2^{s+t} + 2^s - 3$  in  $P_4$  then  $\omega(x) = (3^{(s)}, 2^{(t)}, 1^{(u)})$ .*

*Proof.* Observe that  $z = x_1^{2^{s+t+u}-1} x_2^{2^{s+t}-1} x_3^{2^s-1}$  is the minimal spike of degree  $2^{s+t+u} + 2^{s+t} + 2^s - 3$  and  $\omega(z) = (3^{(s)}, 2^{(t)}, 1^{(u)})$ . Since  $2^{s+t+u} + 2^{s+t} + 2^s - 3$  is odd and  $x$  is admissible, using Lemma 2.13, we get  $\omega_i(x) = 3$  for  $1 \leq i \leq s$ . Set  $x' = \prod_{1 \leq i \leq s} X_{I_{i-1}(x)}^{2^{i-1}}$ . Then,  $x = x' y^{2^s}$  for some monomial  $y$ . We have  $\omega_j(y) = \omega_{j+s}(x)$  for all  $j \geq 1$  and  $\deg y = 2^{t+u} + 2^u - 2$ .

Since  $x$  is admissible, using Theorem 2.9, we see that  $y$  is also admissible. By a direct computation we see that if  $w$  is a monomial such that  $\omega(w) = (3, 2, 3)$ , then  $w$  is inadmissible. Combining this fact, Lemma 5.3.1, Proposition 2.10 and Theorem 2.9, we obtain  $\omega(y) = (2^{(t)}, 1^{(u)})$ . The lemma is proved.  $\square$

Applying Theorem 1.3, we obtain the following.

**Proposition 5.6.2.** *Let  $s, t, u$  be positive integers. If  $s \geq 3$ , then  $\Phi(B_3(n))$  is a minimal set of generators for  $\mathcal{A}$ -module  $P_4$  in degree  $n = 2^{s+t+u} + 2^{s+t} + 2^s - 3$ .*

So, we need only to consider the cases  $s = 1$  and  $s = 2$ .

**5.6.1. The subcase  $s = t = 1$ .**

For  $s = 1, t = 1$ , we have  $n = 2^{u+2} + 3$ . According to Theorem 4.3, we have

$$B_3(n) = \begin{cases} \psi(\Phi(B_2(2^{u+1}))), & \text{if } u \neq 2, \\ \psi(\Phi(B_2(8))) \cup \{x_1^7 x_2^9 x_3^3\}, & \text{if } u = 2. \end{cases}$$

**Proposition 5.6.3.**

i)  $C_4(11) = \Phi(B_3(11)) \cup \{x_1^3 x_2^4 x_3 x_4^3, x_1^3 x_2^4 x_3^3 x_4\}$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree 11.

ii)  $C_4(19) = \Phi(B_3(19)) \cup \{x_1^7 x_2^9 x_3^2 x_4, x_1^3 x_2^{12} x_3 x_4^3, x_1^3 x_2^{12} x_3^3 x_4, x_1^3 x_2^4 x_3 x_4^{11}, x_1^3 x_2^4 x_3^{11} x_4, x_1^3 x_2^7 x_3^8 x_4, x_1^7 x_2^8 x_3 x_4^3, x_1^7 x_2^8 x_3^3 x_4, x_1^3 x_2^4 x_3^3 x_4^9, x_1^3 x_2^4 x_3^9 x_4^3\}$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree 19.

iii)  $C_4(n) = \Phi(B_3(n)) \cup \{x_1^3 x_2^4 x_3 x_4^{2^{u+2}-5}, x_1^3 x_2^4 x_3^{2^{u+2}-5} x_4, x_1^3 x_2^4 x_3^3 x_4^{2^{u+2}-7}\}$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree  $n = 2^{u+2} + 3$ , with any positive integer  $u \geq 3$ .

By a direct computation, we can easily obtain the following lemma.

**Lemma 5.6.4.** *The following monomials are strictly inadmissible:*

- i)  $x_1^3 x_2^4 x_3^4 x_4 x_i x_j^3$ ,  $i, j > 1$ ,  $i \neq j$ ,  $x_1^7 x_2^3 x_3^4 x_4 x_j$ ,  $x_1^3 x_2^5 x_3^5 x_4 x_j$ ,  $j = 3, 4$ .
- ii)  $X_2 x_1^2 x_j^2 x_2^{28}$ ,  $X_j x_1^2 x_2^4 x_3^{24}$ ,  $X_2 x_1^2 x_j^2 x_2^8 x_3^{16}$ ,  $X_j x_1^2 x_2^4 x_3^8 x_4^{18}$ ,  $X_j x_1^2 x_2^4 x_3^{10} x_4^{16}$ ,  $X_j x_1^2 x_2^2 x_3^8 x_4^{16}$ ,  $X_3 x_1^2 x_2^2 x_3^4 x_4^{24}$ ,  $X_2 x_1^2 x_2^4 x_3^{24}$ ,  $i = 1, 2$ ,  $j = 3, 4$ .

*Proof of Theorem 5.6.3.* Let  $x$  be an admissible monomial of degree  $n = 2^{u+2} + 3$  in  $P_4$ . By Lemma 5.6.1,  $\omega_1(x) = 3$ . So,  $x = X_i y^2$  with  $y$  a monomial of degree  $2^{u+1}$ . Since  $x$  is admissible, by Theorem 2.9,  $y \in B_4(2^{u+1})$ . By a direct computation, we see that if  $x = X_i y^2$  with  $y \in B_4(2^{u+1})$  and  $x$  does not belong to the set  $C_4(n)$  as given in the proposition, then there is a monomial  $w$  which is given in one of Lemmas 5.3.3, 5.6.4 such that  $x = w y^{2^r}$  for some monomial  $y$  and integer  $r > 1$ . By Theorem 2.9,  $x$  is inadmissible. Hence,  $(QP_4)_n$  is spanned by the set  $[C_4(n)]$ .

Now, we prove that set  $[E(s) \cup \Phi^0(B_3(n))]$  is linearly independent in  $QP_4$ .

Set  $|C_4(2^{u+2} + 3) \cap P_4^+| = m(u)$ , where  $m(1) = 32$ ,  $m(2) = 80$ ,  $m(u) = 64$  for all  $u > 2$ . Suppose that there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{m(u)} \gamma_i d_i = 0, \quad (5.6.4.1)$$

with  $\gamma_i \in \mathbb{F}_2$  and  $d_i = d_{n,i}$ . By a direct computation from the relations  $p_{(j;J)}(\mathcal{S}) \equiv 0$  with  $(j; J) \in \mathcal{N}_4$ , we obtain  $\gamma_i = 0$  for all  $i$  if  $u \neq 2$ .

For  $u = 2$ ,  $\gamma_j = 0$  for  $j = 1, 3, 4, 6, 7, 8, 9, 10, 11, 12, 14, 16, 17, 18, 19, 21, 23, 26, 27, 28, 29, 30, 31, 32, 35, 36, 38, 40, 43, 45, 51, 54, 55, 60, 61, 62, 68, 71, 79, 80$ , and  $\gamma_2 = \gamma_i, i = 5, 24, 25, 41, 42, 52, 53$ ,  $\gamma_{13} = \gamma_i, i = 13, 33, 20, 56, 48, 58$ ,  $\gamma_{15} = \gamma_i, i = 22, 34, 49, 57, 59$ ,  $\gamma_{37} = \gamma_i, i = 67, 70, 75$ ,  $\gamma_{46} = \gamma_i, i = 69, 72, 76$ ,  $\gamma_{65} = \gamma_i, i = 66, 73, 74, 77, 78$ ,  $\gamma_{46} = \gamma_{39} + \gamma_2$ ,  $\gamma_{44} = \gamma_{37} + \gamma_2$ ,  $\gamma_{65} = \gamma_{47} + \gamma_{13}$ ,  $\gamma_{65} = \gamma_{50} + \gamma_{22}$ ,  $\gamma_{63} = \gamma_{37} + \gamma_{13}$ ,  $\gamma_{64} = \gamma_{46} + \gamma_{22}$ .

Substituting the above equalities into the relation (5.6.4.1), we have

$$\gamma_{37}[\theta_1] + \gamma_{46}[\theta_2] + \gamma_{13}[\theta_3] + \gamma_{22}[\theta_4] + \gamma_{65}[\theta_5] + \gamma_2[\theta_6] = 0, \quad (5.6.4.2)$$

where

$$\begin{aligned}\theta_1 &= d_{37} + d_{44} + d_{63} + d_{67} + d_{70} + d_{75}, \\ \theta_2 &= d_{39} + d_{46} + d_{64} + d_{69} + d_{72} + d_{76}, \\ \theta_3 &= d_{13} + d_{20} + d_{33} + d_{47} + d_{48} + d_{56} + d_{58} + d_{63}, \\ \theta_4 &= d_{15} + d_{22} + d_{34} + d_{49} + d_{50} + d_{57} + d_{59} + d_{64}, \\ \theta_5 &= d_{47} + d_{50} + d_{65} + d_{66} + d_{73} + d_{74} + d_{77} + d_{78}, \\ \theta_6 &= d_2 + d_5 + d_{24} + d_{25} + d_{39} + d_{41} + d_{42} + d_{44} + d_{52} + d_{53}.\end{aligned}$$

We need to prove  $\gamma_2 = \gamma_{13} = \gamma_{22} = \gamma_{37} = \gamma_{46} = \gamma_{65} = 0$ . The proof is divided into 4 steps.

*Step 1.* First we prove  $\gamma_{65} = 0$  by showing the polynomial  $[\theta] = [\beta_1\theta_1 + \beta_2\theta_2 + \beta_3\theta_3 + \beta_4\theta_4 + \theta_5 + \beta_6\theta_6] \neq 0$  for all  $\beta_1, \beta_2, \beta_3, \beta_4, \beta_6 \in \mathbb{F}_2$ . Suppose the contrary that this polynomial is hit. Then we have

$$\theta = Sq^1(A) + Sq^2(B) + Sq^4(C) + Sq^8(D),$$

for some polynomials  $A, B, C, D$  in  $P_4^+$ . Let  $(Sq^2)^3$  act on the both sides of this equality. Using the relations  $(Sq^2)^3Sq^1 = 0, (Sq^2)^3Sq^2 = 0$ , we get

$$(Sq^2)^3(\theta) = (Sq^2)^3Sq^4(C) + (Sq^2)^3Sq^8(D).$$

The monomial  $x_1^7x_2^{12}x_3^4x_4^2$  is a term of  $(Sq^2)^3(\theta)$ . If  $x_1^7x_2^{12}x_3^4x_4^2$  is a term of the polynomial  $(Sq^2)^3Sq^8(y)$  with  $y$  a monomial of degree 11 in  $P_4$ , then  $y = x_1^7f_1(z)$  with  $z$  a monomial of degree 4 in  $P_3$ . Then  $x_1^7x_2^{12}x_3^4x_4^2$  is a term of  $x_1^7(Sq^2)^3Sq^8(f_1(z)) = 0$ . This is a contradiction. So,  $x_1^7x_2^{12}x_3^4x_4^2$  is not a term of  $(Sq^2)^3Sq^8(D)$  for all  $D$ . Hence,  $x_1^7x_2^{12}x_3^4x_4^2$  is a term of  $(Sq^2)^3Sq^4(C)$ , then either  $x_1^7x_2^5x_3x_4^2$  or  $x_1^7x_2^5x_3^2x_4$  or  $x_1^7x_2^6x_3x_4$  is a term of  $C$ .

Suppose  $x_1^7x_2^5x_3^2x_4$  is a term of  $C$ . Then

$$(Sq^2)^3(\theta + Sq^4(x_1^7x_2^5x_3^2x_4)) = (Sq^2)^3(Sq^4(C') + Sq^8(D)),$$

where  $C' = C + x_1^7x_2^5x_3^2x_4$ . We see that the monomial  $x_1^{16}x_2^6x_3^2x_4$  is a term of  $(Sq^2)^3(\theta + Sq^4(x_1^7x_2^5x_3^2x_4))$ . This monomial is not a term of  $(Sq^2)^3Sq^8(D)$  for all  $D$ . So, it is a term of  $(Sq^2)^3Sq^4(C')$ . Then either  $x_1^7x_2^5x_3^2x_4$  or  $x_1^7x_2^6x_3x_4$  is a term of  $C$ . Since  $x_1^7x_2^5x_3^2x_4$  is not a term of  $C'$ ,  $x_1^7x_2^6x_3x_4$  is a term of  $C'$ . Hence, we obtain

$$(Sq^2)^3(\theta + Sq^4(x_1^7x_2^5x_3^2x_4 + x_1^7x_2^6x_3x_4)) = (Sq^2)^3(Sq^4(C'') + Sq^8(D)),$$

where  $C'' = C + x_1^7x_2^5x_3^2x_4 + x_1^7x_2^6x_3x_4$ . Now  $x_1^7x_2^{12}x_3^4x_4^2$  is a term of

$$(Sq^2)^3(\theta + Sq^4(x_1^7x_2^5x_3^2x_4 + x_1^7x_2^6x_3x_4))$$

So, either  $x_1^7x_2^5x_3x_4^2$  or  $x_1^7x_2^5x_3^2x_4$  or  $x_1^7x_2^6x_3x_4$  is a term of  $C''$ . Since  $x_1^7x_2^5x_3^2x_4 + x_1^7x_2^6x_3x_4$  is a summand of  $C''$ ,  $x_1^7x_2^5x_3x_4^2$  is a term of  $C''$ . Then  $x_1^{16}x_2^6x_3^2x_4$  is a term of  $(Sq^2)^3(\theta + Sq^4(x_1^7x_2^5x_3^2x_4 + x_1^7x_2^5x_3x_4^2 + x_1^7x_2^6x_3x_4))$ . So, either  $x_1^7x_2^5x_3x_4^2$  or  $x_1^7x_2^5x_3^2x_4$  or  $x_1^7x_2^6x_3x_4$  is a term of  $C'' + x_1^7x_2^5x_3x_4^2$  and we have a contradiction.

By a same argument, if either  $x_1^7x_2^5x_3x_4^2$  or  $x_1^7x_2^6x_3x_4$  is a term of  $C$  then we have also a contradiction. Hence,  $[\theta] \neq 0$  and  $\gamma_{65} = 0$ .

*Step 2.* By a direct computation, we see that the homomorphism  $\varphi_3$  sends (5.6.4.2) to

$$\gamma_{37}[\theta_1] + \gamma_2[\theta_3] + \gamma_{22}[\theta_4] + \gamma_{46}[\theta_5] + \gamma_{13}[\theta_6] = 0.$$

By Step 1, we obtain  $\gamma_{46} = 0$ .

Step 3. The homomorphism  $\varphi_2$  sends (5.6.4.2) to

$$\gamma_{13}[\theta_1] + \gamma_{22}[\theta_2] + \gamma_{37}[\theta_3] + \gamma_2[\theta_6] = 0.$$

By Step 2, we obtain  $\gamma_{22} = 0$ .

Step 4. Now the homomorphism  $\varphi_3$  sends (5.6.4.2) to  $\gamma_{37}[\theta_2] + \gamma_{13}[\theta_4] + \gamma_2[\theta_6] = 0$ . Combining Step 2 and Step 3, we obtain  $\gamma_{13} = \gamma_{37} = 0$ .

Since  $\varphi_2([\theta_3]) = [\theta_6]$ , we get  $\gamma_2 = 0$ . So, we obtain  $\gamma_j = 0$  for all  $j$ . The proposition follows.  $\square$

### 5.6.2. The subcase $s = 1, t = 2$ .

For  $s = 1, t = 2$ , we have  $n = 2^{u+3} + 7 = 2m + 3$  with  $m = 2^{u+2} + 2$ . Combining Theorem 1.3 and Theorem 4.3, we have  $B_3(n) = \psi(\Phi(B_2(m)))$ , where

$$B_2(m) = \begin{cases} \{x_1^3 x_2^7, x_1^7 x_2^3\}, & \text{if } u = 1, \\ \{x_1^3 x_2^{2^{u+2}-1}, x_1^{2^{u+2}-1} x_2^3, x_1^7 x_2^{2^{u+2}-5}\}, & \text{if } u > 1. \end{cases}$$

Denote by  $F(u)$  the set of all the following monomials:

$$\begin{aligned} & x_1^3 x_2^4 x_3 x_4^{2^{u+3}-1}, x_1^3 x_2^4 x_3^{2^{u+3}-1} x_4, x_1^3 x_2^{2^{u+3}-1} x_3^4 x_4, x_1^{2^{u+3}-1} x_2^3 x_3^4 x_4, \\ & x_1^3 x_2^7 x_3^{2^{u+3}-4} x_4, x_1^7 x_2^3 x_3^{2^{u+3}-4} x_4, x_1^7 x_2^{2^{u+3}-5} x_3^4 x_4, x_1^7 x_2^7 x_3^{2^{u+3}-8} x_4, \\ & x_1^3 x_2^4 x_3^3 x_4^{2^{u+3}-3}, x_1^3 x_2^4 x_3^{2^{u+3}-5} x_4^5, x_1^3 x_2^4 x_3^7 x_4^{2^{u+3}-7}, x_1^3 x_2^7 x_3^4 x_4^{2^{u+3}-7}, \\ & x_1^7 x_2^3 x_3^4 x_4^{2^{u+3}-7}, x_1^3 x_2^7 x_3^8 x_4^{2^{u+3}-11}, x_1^7 x_2^3 x_3^8 x_4^{2^{u+3}-11}. \end{aligned}$$

### Proposition 5.6.5.

- i)  $C_4(23) = \Phi(B_3(23)) \cup F(1) \cup \{x_1^7 x_2^9 x_3^2 x_4^5, x_1^7 x_2^9 x_3^3 x_4^4\}$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree 23.
- ii)  $C_4(n) = \Phi(B_3(n)) \cup F(u) \cup \{x_1^7 x_2^7 x_3^8 x_4^{2^{u+3}-15}, x_1^7 x_2^7 x_3^9 x_4^{2^{u+3}-16}, x_1^3 x_2^4 x_3^{11} x_4^{2^{u+3}-11}\}$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree  $n = 2^{u+3} + 7$  with any positive integer  $u > 1$ .

By a direct computation, we can easy obtain the following lemma.

**Lemma 5.6.6.** *The following monomials are strictly inadmissible:*

- i)  $X_2 x_1^2 x_j^6 x_2^{12}, X_j x_1^2 x_2^4 x_3^8 x_4^6, X_2 x_1^2 x_i^4 x_2^8 x_3^4 x_4^6, X_2 x_1^2 x_2^4 x_3^8 x_4^6, i = 1, 2, j = 3, 4$ .
- ii)  $X_3 x_1^2 x_2^2 x_i^{12} x_3^{20}, X_3 x_1^2 x_2^2 x_i^4 x_3^{20} x_4^4, X_j x_1^2 x_2^2 x_i^{12} x_3^4 x_4^{16}, X_j x_1^2 x_2^4 x_i^{14} x_3^{16},$   
 $X_j x_1^6 x_2^{10} x_3^4 x_4^{16}, X_j x_1^6 x_2^{10} x_3^{16} x_4^4, X_3 x_1^6 x_2^{10} x_3^{20}, X_2 x_1^2 x_2^4 x_3^{14} x_4^{16}, i = 1, 2, j = 3, 4$ .

*Proof of Proposition 5.6.5.* Let  $x$  be an admissible monomial of degree  $n = 2^{u+3} + 7$  in  $P_4$ . By Lemma 5.6.1,  $\omega_1(x) = 3$ . So,  $x = X_i y^2$  with  $y$  a monomial of degree  $2^{u+2} + 2$ . Since  $x$  is admissible, by Theorem 2.9,  $y \in B_4(2^{u+2} + 2)$ .

By a direct computation, we see that if  $x = X_i y^2$  with  $y \in B_4(2^{u+2} + 2)$  and  $x$  does not belong to the set  $C_4(n)$  as given in the proposition, then there is a monomial  $w$  which is given in one of Lemmas 5.6.6, 5.3.3 such that  $x = w y^{2^r}$  for some monomial  $y$  and integer  $r > 1$ . By Theorem 2.9,  $x$  is inadmissible. Hence,  $(QP_4)_n$  is spanned by the set  $[C_4(n)]$ .

Now, we prove that set  $[C_4(n)]$  is linearly independent in  $QP_4$ .

For  $u = 1$ , we have,  $|C_4^+(23) \cap P_4^+| = 99$ . Suppose that there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{99} \gamma_i d_i = 0, \quad (5.6.6.1)$$

with  $\gamma_i \in \mathbb{F}_2$  and  $d_i = d_{23,i}$ . By a direct computation from the relations  $p_{(j;J)}(\mathcal{S}) \equiv 0$  with  $(j;J) \in \mathcal{N}_4$ , we obtain  $\gamma_i = 0$  for all  $i \in E$ , with some  $E \subset \mathbb{N}_{99}$  and the relation (5.6.4.2) becomes

$$\sum_{i=1}^{15} c_i [\theta_i] = 0, \quad (5.6.6.2)$$

where  $c_1 = \gamma_1, c_2 = \gamma_4, c_3 = \gamma_{33}, c_4 = \gamma_{94}, c_5 = \gamma_2, c_6 = \gamma_{22}, c_7 = \gamma_{74}, c_8 = \gamma_{29}, c_9 = \gamma_{81}, c_{10} = \gamma_{68}, c_{11} = \gamma_{10}, c_{12} = \gamma_{43}, c_{13} = \gamma_{54}, c_{14} = \gamma_{70}, c_{15} = \gamma_{11}$  and

$$\begin{aligned} \theta_1 &= d_1 + d_{17} + d_{37} + d_{49}, \\ \theta_2 &= d_4 + d_{21} + d_{44} + d_{53}, \\ \theta_3 &= d_{33} + d_{36} + d_{72} + d_{73}, \\ \theta_4 &= d_{94} + d_{97} + d_{98} + d_{99}, \\ \theta_5 &= d_2 + d_{19} + d_{40} + d_{51}, \\ \theta_6 &= d_{22} + d_{25} + d_{62} + d_{63}, \\ \theta_7 &= d_{74} + d_{77} + d_{82} + d_{83}, \\ \theta_8 &= d_{12} + d_{14} + d_{26} + d_{29} + d_{66} + d_{67}, \\ \theta_9 &= d_{40} + d_{42} + d_{78} + d_{81} + d_{86} + d_{87}, \\ \theta_{10} &= d_{10} + d_{15} + d_{24} + d_{27} + d_{46} + d_{47} + d_{64} + d_{65}, \\ \theta_{11} &= d_{38} + d_{43} + d_{46} + d_{47} + d_{76} + d_{79} + d_{84} + d_{85}, \\ \theta_{12} &= d_{62} + d_{67} + d_{68} + d_{71} + d_{88} + d_{89} + d_{92} + d_{93}, \\ \theta_{13} &= d_{47} + d_{54} + d_{57} + d_{62} + d_{69} + d_{82} + d_{85} + d_{88} + d_{90}, \\ \theta_{14} &= d_{12} + d_{15} + d_{19} + d_{20} + d_{46} + d_{47} + d_{51} + d_{52} + d_{58} + d_{61} \\ &\quad + d_{64} + d_{66} + d_{67} + d_{70} + d_{84} + d_{87} + d_{89} + d_{91}, \\ \theta_{15} &= d_{11} + d_{12} + d_{18} + d_{20} + d_{24} + d_{25} + d_{26} + d_{27} + d_{38} + d_{40} + d_{45} \\ &\quad + d_{47} + d_{48} + d_{50} + d_{52} + d_{57} + d_{61} + d_{63} + d_{64} + d_{65} + d_{66} \\ &\quad + d_{67} + d_{69} + d_{77} + d_{78} + d_{83} + d_{85} + d_{86} + d_{87} + d_{89} + d_{90}. \end{aligned}$$

Now, we show that  $c_i = 0$  for  $i = 1, 2, \dots, 15$ . The proof is divided into 6 steps.

*Step 1.* Set  $\theta = \theta_1 + \sum_{i=2}^{15} \beta_i \theta_i$  for  $\beta_i \in \mathbb{F}_2, i = 2, 3, \dots, 15$ . We prove that  $[\theta] \neq 0$ . Suppose the contrary that  $\theta$  is hit. Then we have

$$\theta = Sq^1(A) + Sq^2(B) + Sq^4(C) + Sq^8(D)$$

for some polynomials  $A, B, C, D \in P_4^+$ . Let  $(Sq^2)^3$  act to the both sides of the above equality, we obtain

$$(Sq^2)^3(\theta) = (Sq^2)^3 Sq^4(C) + (Sq^2)^3 Sq^8(D).$$

By a similar computation as in the proof of Proposition 5.4.5, we see that the monomial  $x_1^8 x_2^4 x_3^2 x_4^{15}$  is a term of  $(Sq^2)^3(\theta)$ . This monomial is not a term of  $(Sq^2)^3(Sq^4(C) + Sq^8(D))$  for all polynomials  $C, D$  and we have a contradiction. So,  $[\theta] \neq 0$  and we get  $c_1 = \gamma_1 = 0$ .

By an argument analogous to the previous one, we get  $c_2 = c_3 = c_4 = 0$ . Now, the relation (5.6.6.2) becomes

$$\sum_{i=5}^{15} c_i [\theta_i] = 0. \quad (5.6.6.3)$$

*Step 2.* The homomorphisms

$$\varphi_1, \varphi_1\varphi_3, \varphi_1\varphi_3\varphi_4, \varphi_1\varphi_3\varphi_2, \varphi_1\varphi_3\varphi_2\varphi_4, \varphi_1\varphi_3\varphi_4\varphi_2\varphi_3$$

send (5.6.6.3) respectively to

$$\begin{aligned} c_{10}[\theta_3] &= 0 \pmod{\langle [\theta_5], [\theta_6], \dots, [\theta_{15}] \rangle}, \\ c_9[\theta_3] &= 0 \pmod{\langle [\theta_5], [\theta_6], \dots, [\theta_{15}] \rangle}, \\ c_7[\theta_3] &= 0 \pmod{\langle [\theta_5], [\theta_6], \dots, [\theta_{15}] \rangle}, \\ c_8[\theta_3] &= 0 \pmod{\langle [\theta_5], [\theta_6], \dots, [\theta_{15}] \rangle}, \\ c_6[\theta_3] &= 0 \pmod{\langle [\theta_5], [\theta_6], \dots, [\theta_{15}] \rangle}, \\ c_5[\theta_3] &= 0 \pmod{\langle [\theta_5], [\theta_6], \dots, [\theta_{15}] \rangle}. \end{aligned}$$

Using the results in Step 1, we get  $c_5 = c_6 = c_7 = c_8 = c_9 = c_{10} = 0$ . So, the relation (5.6.6.3) becomes

$$c_{11}[\theta_{11}] + c_{12}[\theta_{12}] + c_{13}[\theta_{13}] + c_{14}[\theta_{14}] + c_{15}[\theta_{15}] = 0. \quad (5.6.6.4)$$

*Step 3.* The homomorphism  $\varphi_1$  sends (5.6.6.4) to

$$\begin{aligned} &c_{13}[\theta_6] + (c_{14} + c_{15})[\theta_7] + (c_{11} + c_{12})[\theta_{11}] \\ &+ c_{12}[\theta_{12}] + c_{13}[\theta_{13}] + c_{14}[\theta_{14}] + c_{15}[\theta_{15}] = 0. \end{aligned}$$

By Step 2, we get  $c_{13} = 0$  and  $c_{14} = c_{15}$ . So, the relation (5.6.6.4) becomes

$$c_{11}[\theta_{11}] + c_{12}[\theta_{12}] + c_{14}[\theta_{14}] + c_{14}[\theta_{15}] = 0. \quad (5.6.6.5)$$

*Step 4.* The homomorphism  $\varphi_3$  sends (5.6.6.5) to

$$c_{11}[\theta_{11}] + c_{14}[\theta_{12}] + (c_{12} + c_{14})[\theta_{13}] + c_{14}[\theta_{14}] + c_{14}[\theta_{15}] = 0.$$

By Step 3, we get  $c_{12} = c_{14}$ . Then the relation (5.6.6.5) becomes

$$c_{11}[\theta_{11}] + c_{12}[\theta_{12}] + c_{12}[\theta_{14}] + c_{12}[\theta_{15}] = 0. \quad (5.6.6.6)$$

*Step 5.* The homomorphism  $\varphi_2$  sends (5.6.6.6) to

$$(c_{11} + c_{12})[\theta_{12}] + c_{12}[\theta_{14}] + c_{12}[\theta_{15}] = 0.$$

From the result in Step 4, we get  $c_{11} = 0$ . Then the relation (5.6.6.6) becomes

$$c_{12}([\theta_{12}] + [\theta_{14}] + [\theta_{15}]) = 0. \quad (5.6.6.7)$$

*Step 6.* The homomorphism  $\varphi_1$  sends (5.6.6.7) to

$$c_{12}[\theta_{11}] + c_{12}([\theta_{12}] + [\theta_{14}] + [\theta_{15}]) = 0.$$

By the result in Step 5, we have  $c_{12} = 0$ . The case  $u = 1$  of the proposition is completely proved.

For  $u > 1$ , we have  $|C_4(n)^+| = 141$ . Suppose that there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{141} \gamma_i d_i = 0, \quad (5.6.6.8)$$

with  $\gamma_i \in \mathbb{F}_2$  and  $d_i = d_{n,i} \in B_4^+(n)$ . By a direct computation from the relations  $p_{(j;J)}(\mathcal{S}) \equiv 0$  with  $(j;J) \in \mathcal{N}_4$ , we obtain  $\gamma_i = 0$  for all  $i \notin E$ , with some  $E \subset \mathbb{N}_{141}$  and the relation (5.6.6.8) becomes

$$\sum_{i=1}^{15} c_i[\theta_i] = 0, \quad (5.6.6.9)$$



where  $c_1 = \gamma_1, c_2 = \gamma_6, c_3 = \gamma_{51}, c_4 = \gamma_{136}, c_5 = \gamma_2, c_6 = \gamma_{31}, c_7 = \gamma_{107}, c_8 = \gamma_{40}, c_9 = \gamma_{116}, c_{10} = \gamma_{101}, c_{11} = \gamma_{14}, c_{12} = \gamma_{56}, c_{13} = \gamma_{79}, c_{14} = \gamma_{23}, c_{15} = \gamma_{15}$  and

$$\begin{aligned}
\theta_1 &= d_1 + d_{25} + d_{55} + d_{73}, \\
\theta_2 &= d_6 + d_{30} + d_{66} + d_{78}, \\
\theta_3 &= d_{51} + d_{54} + d_{105} + d_{106}, \\
\theta_4 &= d_7 + d_8 + d_{47} + d_{48}, \\
\theta_5 &= d_2 + d_{27} + d_{58} + d_{75}, \\
\theta_6 &= d_{31} + d_{34} + d_{89} + d_{90}, \\
\theta_7 &= d_{107} + d_{110} + d_{117} + d_{118}, \\
\theta_8 &= d_{16} + d_{22} + d_{35} + d_{40} + d_{94} + d_{95}, \\
\theta_9 &= d_{58} + d_{64} + d_{111} + d_{116} + d_{122} + d_{123}, \\
\theta_{10} &= d_{89} + d_{95} + d_{101} + d_{104} + d_{124} + d_{127} + d_{129} + d_{130}, \\
\theta_{11} &= d_{14} + d_{19} + d_{33} + d_{36} + d_{68} + d_{69} + d_{91} + d_{92}, \\
\theta_{12} &= d_{56} + d_{61} + d_{68} + d_{69} + d_{109} + d_{112} + d_{119} + d_{120}, \\
\theta_{13} &= d_{67} + d_{69} + d_{79} + d_{82} + d_{89} + d_{90} + d_{117} + d_{118} + d_{124} + d_{125}, \\
\theta_{14} &= d_{16} + d_{23} + d_{27} + d_{29} + d_{70} + d_{71} + d_{72} + d_{75} + d_{77} \\
&\quad + d_{83} + d_{88} + d_{94} + d_{95} + d_{122} + d_{123} + d_{126} + d_{127}, \\
\theta_{15} &= d_{15} + d_{19} + d_{26} + d_{27} + d_{33} + d_{34} + d_{35} + d_{36} + d_{58} \\
&\quad + d_{61} + d_{68} + d_{69} + d_{70} + d_{74} + d_{75} + d_{82} + d_{83} + d_{91} \\
&\quad + d_{92} + d_{109} + d_{110} + d_{111} + d_{112} + d_{119} + d_{120} + d_{125}.
\end{aligned}$$

Now, we prove  $c_i = 0$  for  $i = 1, 2, \dots, 15$ . The proof is divided into 6 steps.

*Step 1.* First, we prove  $c_1 = 0$ . Set  $\theta = \theta_1 + \sum_{j=2}^{15} c_j \theta_j$ . We show that  $[\theta] \neq 0$  for all  $c_j \in \mathbb{F}_2, j = 2, 3, \dots, 15$ . Suppose the contrary that  $\theta$  is hit. Then we have

$$\theta = \sum_{m=0}^{u+2} Sq^{2^m}(A_m),$$

for some polynomials  $A_m, m = 0, 1, \dots, u+2$ . Let  $(Sq^2)^3$  act on the both sides of this equality. Since  $(Sq^2)^3 Sq^1 = 0, (Sq^2)^3 Sq^2 = 0$ , we get

$$(Sq^2)^3(\theta) = \sum_{m=2}^{u+2} (Sq^2)^3 Sq^{2^m}(A_m).$$

It is easy to see that the monomial  $x = x_1^8 x_2^4 x_3^2 x_4^{2^{u+3}-1}$  is a term of  $(Sq^2)^3(\theta)$ . Hence, it is a term of  $(Sq^2)^3 Sq^{2^m}(y)$  for some monomial  $y$  of degree  $2^{u+3} - 2^m + 7$  with  $m \geq 2$ . Then  $y = x_2^{2^{u+3}-1} f_2(z)$  with  $z$  a monomial of degree  $8 - 2^m \leq 4$  in  $P_3$  and  $x$  is a term of  $x_2^{2^{u+3}-1} (Sq^2)^3 Sq^{2^m}(z)$ . If  $m > 2$  then  $Sq^{2^m}(z) = 0$ . If  $m = 2$ , then  $Sq^{2^2}(z) = z^2$ . Hence,  $(Sq^2)^3 Sq^{2^m}(z) = (Sq^2)^3(z^2) = 0$ . So,  $x$  is not a term of

$$(Sq^2)^3(\theta) = \sum_{m=2}^{u+2} (Sq^2)^3 Sq^{2^m}(A_m),$$

for all polynomial  $A_m$  with  $m > 1$ . This is a contradiction. So, we get  $c_1 = 0$ .

By an argument analogous to the previous one, we get  $c_2 = c_3 = c_4 = 0$ . Then the relation (5.6.6.9) becomes

$$\sum_{i=5}^{15} c_i[\theta_i] = 0. \quad (5.6.6.10)$$

*Step 2.* The homomorphisms

$$\varphi_1, \varphi_1\varphi_3, \varphi_1\varphi_3\varphi_4, \varphi_1\varphi_3\varphi_2, \varphi_1\varphi_3\varphi_2\varphi_4, \varphi_1\varphi_3\varphi_4\varphi_2\varphi_3$$

send (5.6.6.3) respectively to

$$\begin{aligned} c_{10}[\theta_3] &= 0 \pmod{\langle [\theta_5], [\theta_6], \dots, [\theta_{15}] \rangle}, \\ c_9[\theta_3] &= 0 \pmod{\langle [\theta_5], [\theta_6], \dots, [\theta_{15}] \rangle}, \\ c_7[\theta_3] &= 0 \pmod{\langle [\theta_5], [\theta_6], \dots, [\theta_{15}] \rangle}, \\ c_8[\theta_3] &= 0 \pmod{\langle [\theta_5], [\theta_6], \dots, [\theta_{15}] \rangle}, \\ c_6[\theta_3] &= 0 \pmod{\langle [\theta_5], [\theta_6], \dots, [\theta_{15}] \rangle}, \\ c_5[\theta_3] &= 0 \pmod{\langle [\theta_5], [\theta_6], \dots, [\theta_{15}] \rangle}. \end{aligned}$$

By Step 1, we get  $c_5 = c_6 = c_7 = c_8 = c_9 = c_{10} = 0$ . So, the relation (5.6.6.3) becomes

$$c_{11}[\theta_{11}] + c_{12}[\theta_{12}] + c_{13}[\theta_{13}] + c_{14}[\theta_{14}] + c_{15}[\theta_{15}] = 0. \quad (5.6.6.11)$$

*Step 3.* Applying the homomorphism  $\varphi_1$  to (5.6.6.11), we get

$$c_{13}[\theta_6] + c_{14}[\theta_8] + (c_{11} + c_{12} + c_{15})[\theta_{11}] + c_{12}[\theta_{12}] + c_{13}[\theta_{13}] + c_{14}[\theta_{14}] + c_{15}[\theta_{15}] = 0.$$

By the results in Step 2, we obtain  $c_{13} = c_{14} = 0$ . Then the relation (5.6.6.11) becomes

$$c_{11}[\theta_{11}] + c_{12}[\theta_{12}] + c_{14}[\theta_{15}] = 0. \quad (5.6.6.12)$$

*Step 4.* Applying the homomorphism  $\varphi_3$  to the relation (5.6.6.12) we obtain

$$c_{11}[\theta_{11}] + c_{12}[\theta_{13}] + c_{15}[\theta_{15}] = 0.$$

By the results in Step 3, we get  $c_{12} = 0$ . So, the relation (5.6.6.12) becomes

$$c_{11}[\theta_{11}] + c_{15}[\theta_{15}] = 0. \quad (5.6.6.13)$$

*Step 5.* Applying the homomorphism  $\varphi_2$  to the relation (5.6.6.12) one gets

$$c_{11}[\theta_{13}] + c_{15}[\theta_{15}] = 0.$$

By Step 4, we get  $c_{10} = \gamma_{41} = 0$ . So, the relation (5.6.6.13) becomes

$$c_{15}[\theta_{15}] = 0. \quad (5.6.6.14)$$

*Step 6.* Applying the homomorphism  $\varphi_1$  to the relation (5.6.6.14) we obtain

$$c_{15}[\theta_{11}] + c_{15}[\theta_{15}] = 0.$$

By Step 5, we get  $c_{15}$ . The proposition is completely proved.  $\square$

### 5.6.3. The subcase $s = 1, t > 2$ .

For  $s = 1, t > 2$ , we have  $n = 2^{t+u+1} + 2^{t+1} - 1 = 2m + 3$  with  $m = 2^{t+u} + 2^t - 2$ . From Theorem 4.3, we have  $B_3(n) = \psi(\Phi(B_2(m)))$ .

**Proposition 5.6.7.**

i)  $C_4(n) = \Phi(B_3(n)) \cup \{x_1^3 x_2^4 x_3^{2^{t+1}-5} x_4^{2^{t+2}-3}, x_1^3 x_2^4 x_3^{2^{t+2}-5} x_4^{2^{t+1}-3}\}$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree  $n = 2^{t+2} + 2^{t+1} - 1$  with any positive integer  $t > 2$ .

ii)  $C_4(n) = \Phi(B_3(n)) \cup A(t, u)$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree  $n = 2^{t+u+1} + 2^{t+1} - 1$  with any positive integers  $t > 2, u > 1$ , where  $A(t, u)$  is the set consisting of 3 monomials:

$$x_1^3 x_2^4 x_3^{2^{t+1}-5} x_4^{2^{t+u+1}-3}, x_1^3 x_2^4 x_3^{2^{t+u+1}-5} x_4^{2^{t+1}-3}, x_1^3 x_2^4 x_3^{2^{t+2}-5} x_4^{2^{t+u+1}-2^{t+1}-3}.$$

By a direct computation, we can easy obtain the following lemma.

**Lemma 5.6.8.** *The following monomials are strictly inadmissible:*

$$X_3 x_1^2 x_2^2 x_3^8 x_4^{28} x_i^4, X_3 x_1^2 x_2^2 x_3^8 x_4^{12} x_i^4, i = 1, 2, X_4 x_1^6 x_2^{10} x_3^{12} x_4^{16}.$$

*Proof of Proposition 5.6.7.* Let  $x \in P_4$  be an admissible monomial of degree  $n = 2^{t+u+1} + 2^{t+1} - 1$ . By Lemma 5.6.1,  $\omega_1(x) = 3$ . So,  $x = X_i y^2$  with  $y$  a monomial of degree  $2^{t+u} + 2^t - 2$ . Since  $x$  is admissible, by Theorem 2.9,  $y \in B_4(2^{t+u} + 2^t - 2)$ .

By a direct computation, we see that if  $x = X_i y^2$  with  $y \in B_4(2^{t+u} + 2^t - 2)$  and  $x$  does not belongs to the set  $C_4(n)$  as given in the proposition, then there is a monomial  $w$  which is given in one of Lemmas 5.6.8 and 5.3.3 such that  $x = w y^{2^r}$  for some monomial  $y$  and integer  $r > 1$ . By Theorem 2.9,  $x$  is inadmissible. Hence,  $(QP_4)_n$  is spanned by the set  $[C_4(n)]$ .

Now, we prove that set  $[C_4(n)]$  is linearly independent in  $QP_4$ .

We set  $|C_4(n)^+| = m(t, u)$  with  $m(t, 1) = 84$  and  $m(t, u) = 126$  for  $u > 1$ . Suppose that there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{m(t,u)} \gamma_i d_i = 0,$$

with  $\gamma_i \in \mathbb{F}_2$  and  $d_i = d_{n,i}$ . By a direct computation from the relations  $p_{(j;J)}(\mathcal{S}) \equiv 0$  with  $(j; J) \in \mathcal{N}_4$ , we obtain  $\gamma_i = 0$  for all  $i$ .  $\square$

#### 5.6.4. The subcase $s = 2, t = 1$ .

For  $s = 2, t = 1$ , we have  $n = 2^{u+3} + 9$ . According to Theorem 4.3, we have

$$B_3(n) = \begin{cases} \psi^2(\Phi(B_2(2^{u+1}))), & \text{if } u \neq 2, \\ \psi^2(\Phi(B_2(8))) \cup \{x_1^{15} x_2^{19} x_3^7\}, & \text{if } u = 2. \end{cases}$$

Denote by  $G(u)$  the set of 7 monomials:

$$x_1^3 x_2^7 x_3^{2^{u+3}-5} x_4^4, x_1^7 x_2^3 x_3^{2^{u+3}-5} x_4^4, x_1^7 x_2^{2^{u+3}-5} x_3^3 x_4^4, \\ x_1^3 x_2^7 x_3^7 x_4^{2^{u+3}-8}, x_1^7 x_2^3 x_3^7 x_4^{2^{u+3}-8}, x_1^7 x_2^3 x_3^3 x_4^{2^{u+3}-8}, x_1^7 x_2^7 x_3^3 x_4^{2^{u+3}-8} x_4^3,$$

**Proposition 5.6.9.**

i)  $C_4(25) = \Phi(B_3(25)) \cup G(1) \cup \{x_1^7 x_2^9 x_3^3 x_4^6\}$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree 25.

ii)  $C_4(n) = \Phi(B_3(n)) \cup G(u) \cup H(u)$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree  $n = 2^{u+3} + 9$  with any positive integer  $u > 1$ , where

$H(u)$  is the set consisting of 5 monomials:

$$\begin{aligned} & x_1^3 x_2^7 x_3^{11} x_4^{2^{u+3}-12}, \quad x_1^7 x_2^3 x_3^{11} x_4^{2^{u+3}-12}, \quad x_1^7 x_2^{11} x_3^3 x_4^{2^{u+3}-12}, \\ & x_1^7 x_2^7 x_3^8 x_4^{2^{u+3}-13}, \quad x_1^7 x_2^7 x_3^{11} x_4^{2^{u+3}-16}. \end{aligned}$$

The following is proved by a direct computation.

**Lemma 5.6.10.** *The following monomials are strictly inadmissible:*

- i)  $X_3 X_2^2 x_1^4 x_2^8 x_4^4$ ,  $X_j X_2^2 x_1^4 x_2^8 x_4^4$ ,  $X_3^3 x_i^4 x_3^8 x_4^4$ ,  $X_2^3 x_1^4 x_2^8 x_j^4$ ,  $i = 1, 2$ ,  $j = 3, 4$ .
- ii)  $X_4 X_3^2 x_1^{12} x_2^{16} x_3^4$ ,  $X_4 X_2^2 x_1^4 x_2^{24} x_4^4$ ,  $X_4^3 x_i^{12} x_3^{16} x_4^4$ ,  $X_4 X_2^2 x_1^{12} x_2^{16} x_4^4$ ,  $X_4 X_3^4 x_1^4 x_2^8 x_3^{16}$ ,  $X_j X_2^2 x_1^{12} x_2^{16} x_3^4$ ,  $X_j X_2^2 x_1^{12} x_2^{16} x_4^4$ ,  $X_4 X_2^2 x_1^4 x_2^8 x_4^{20}$ ,  $X_j^3 x_1^4 x_2^8 x_j^{16}$ ,  $X_2^3 x_1^{12} x_2^{16} x_j^4$ ,  $X_4^3 x_i^4 x_3^{12} x_4^{16}$ ,  $X_4^3 x_i^{12} x_3^4 x_4^{16}$ ,  $X_3^3 x_i^{12} x_3^{16} x_4^4$ ,  $X_j^3 x_1^4 x_2^8 x_3^{16} x_4^4$ ,  $X_4 X_2^2 x_1^4 x_2^8 x_3^{16} x_4^4$ ,  $X_4^3 x_1^4 x_2^8 x_3^{16} x_4^4$ ,  $i = 1, 2$ ,  $j = 3, 4$ .

*Proof of Proposition 5.6.9.* Let  $x$  be an admissible monomial of degree  $n = 2^{u+3} + 9$  in  $P_4$ .

By Lemma 5.6.1,  $\omega_1(x) = \omega_2(x) = 3$ . So,  $x = X_i X_j^2 y^4$  with  $y$  a monomial of degree  $2^{u+1}$ . Since  $x$  is admissible, by Theorem 2.9,  $y \in B_4(2^{t+u} + 2^t - 2)$ .

By a direct computation, we see that if  $x = X_i X_j^2 y^4$  with  $y \in B_4(2^{t+u} + 2^t - 2)$  and  $x$  does not belongs to the set  $C_4(n)$  given in the proposition, then there is a monomial  $w$  which is given in one of Lemmas 5.6.10, 5.3.3 such that  $x = w y^{2^r}$  for some monomial  $y$  and integer  $r > 1$ . By Theorem 2.9,  $x$  is inadmissible. Hence,  $(QP_4)_n$  is spanned by the set  $[C_4(n)]$ .

Now, we prove that set  $[C_4(n)]$  is linearly independent in  $QP_4$ .

We denote  $|C_4(n)^+| = m(u)$  with  $m(1) = 88$ ,  $m(2) = 165$  and  $m(u) = 154$  for  $u \geq 3$ . Suppose that there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{m(u)} \gamma_i d_i = 0,$$

with  $\gamma_i \in \mathbb{F}_2$  and  $d_i = d_{n,i}$ . By a direct computation from the relations  $p_{(j;J)}(\mathcal{S}) \equiv 0$  with  $(j; J) \in \mathcal{N}_4$ , we obtain  $\gamma_i = 0$  for all  $i$ .  $\square$

#### 5.6.5. The subcase $s = 2$ , $t \geq 2$ .

For  $s = 2$ ,  $t \geq 2$ , we have  $n = 2^{t+u+2} + 2^{t+2} + 1 = 4m + 9$  with  $m = 2^{t+u} + 2^t - 2$ . From Theorem 1.3, we have

$$B_3(n) = \psi^2(\Phi(B_2(m))).$$

Denote by  $B(t, u)$  the set of 8 monomials:

$$\begin{aligned} & x_1^3 x_2^7 x_3^{2^{t+2}-5} x_4^{2^{t+u+2}-4}, \quad x_1^7 x_2^3 x_3^{2^{t+2}-5} x_4^{2^{t+u+2}-4}, \quad x_1^7 x_2^{2^{t+2}-5} x_3^3 x_4^{2^{t+u+2}-4}, \\ & x_1^3 x_2^7 x_3^{2^{t+u+2}-5} x_4^{2^{t+2}-4}, \quad x_1^7 x_2^3 x_3^{2^{t+u+2}-5} x_4^{2^{t+2}-4}, \quad x_1^7 x_2^{2^{t+u+2}-5} x_3^3 x_4^{2^{t+2}-4}, \\ & x_1^7 x_2^7 x_3^{2^{t+2}-8} x_4^{2^{t+u+2}-5}, \quad x_1^7 x_2^7 x_3^{2^{t+u+2}-8} x_4^{2^{t+2}-5}, \end{aligned}$$

and by  $C(t, u)$  the set of 4 monomials:

$$\begin{aligned} & x_1^3 x_2^7 x_3^{2^{t+3}-5} x_4^{2^{t+u+2}-2^{t+2}-4}, \quad x_1^7 x_2^3 x_3^{2^{t+3}-5} x_4^{2^{t+u+2}-2^{t+2}-4}, \\ & x_1^7 x_2^{2^{t+3}-5} x_3^3 x_4^{2^{t+u+2}-2^{t+2}-4}, \quad x_1^7 x_2^7 x_3^{2^{t+3}-8} x_4^{2^{t+u+2}-2^{t+2}-5}. \end{aligned}$$

**Proposition 5.6.11.**

i)  $C_4(n) = \Phi(B_3(n)) \cup B(t, 1)$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree  $n = 2^{t+3} + 2^{t+2} + 1$ .

ii) For any positive integer  $t, u > 1$ ,  $C_4(n) = \Phi(B_3(n)) \cup B(t, u) \cup C(t, u)$  is the set of all the admissible monomials for  $\mathcal{A}$ -module  $P_4$  in degree  $n = 2^{t+u+2} + 2^{t+2} + 1$ .

By a direct computation, we get the following.

**Lemma 5.6.12.** *The following monomials are strictly inadmissible:*

$$X_j X_3^2 x_1^{12} x_2^{12} x_3^{16}, X_4^3 x_i^{12} x_3^{12} x_4^{16}, X_4^3 x_1^{12} x_2^{12} x_4^{16}, X_4^3 x_1^4 x_2^4 x_3^8 x_4^8 x_j^{16}, X_4 X_3^2 x_3^4 x_1^{12} x_4^8 x_2^{16}, \\ X_4 X_3^2 x_1^4 x_2^4 x_4^8 x_i^{16}, X_j^3 x_1^4 x_2^4 x_3^8 x_i^{16}, X_4^3 x_1^4 x_3^4 x_2^8 x_3^8 x_4^{16}, \quad i = 1, 2, \quad j = 3, 4.$$

*Proof of Proposition 5.6.11.* Let  $x \in P_4$  be an admissible monomial of degree  $n = 2^{t+u+2} + 2^{t+2} + 1$ . By Lemma 5.6.1,  $\omega_1(x) = \omega_2(x) = 3$ . So,  $x = X_i X_j^2 y^4$  with  $y$  a monomial of degree  $2^{t+u} + 2^t - 2$ .

Since  $x$  is admissible, by Theorem 2.9,  $y \in B_4(2^{t+u} + 2^t - 2)$ . By a direct computation, we see that if  $x = X_i X_j^2 y^4$  with  $y \in B_4(2^{t+u} + 2^t - 2)$  and  $x$  does not belongs to the set  $C_4(n)$  as given in the proposition, then there is a monomial  $w$  which is given in one of Lemmas 5.6.12, 5.1.3 such that  $x = w y^{2^r}$  for some monomial  $y$  and integer  $r > 1$ . By Theorem 2.9,  $x$  is inadmissible. Hence,  $(QP_4)_n$  is spanned by the set  $[C_4(n)]$ .

Now, we prove that set  $[C_4(n)]$  is linearly independent in  $QP_4$ .

We set  $|C_4(n)^+| = m(t, u)$  with  $m(t, 1) = 154$  and  $m(t, u) = 231$  for  $t \geq 2$ . Suppose that there is a linear relation

$$\mathcal{S} = \sum_{i=1}^{m(t,u)} \gamma_i d_i = 0,$$

with  $\gamma_i \in \mathbb{F}_2$  and  $d_i = d_{n,i}$ . By a direct computation from the relations  $p_{(j;J)}(\mathcal{S}) \equiv 0$  with  $(j; J) \in \mathcal{N}_4$ , we obtain  $\gamma_i = 0$  for all  $i$ .  $\square$

Theorem 1.4 follows from the results in Subsections 5.1-5.6.

**Acknowledgment.** I would like to thank Prof. Nguyễn H. V. Hưng for helpful suggestions and constant encouragement. My thanks also go to all colleagues at the Department of Mathematics, Quy Nhơn University for many conversations.

The original version of this work was completed while the author was visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM) in July, 2013. He would like to thank the VIASM for supporting the visit and hospitality. The work was also supported in part by the Research Project Grant No. B2013.28.129.

I would like to express my warmest thanks to the referees for the careful reading and detailed comments with many helpful suggestions.

## REFERENCES

- [1] J. M. Boardman, *Modular representations on the homology of powers of real projective space*, in: M.C. Tangora (Ed.), *Algebraic Topology*, Oaxtepec, 1991, in: *Contemp. Math.*, vol. 146, 1993, pp. 49-70, MR1224907.
- [2] R. R. Bruner, L. M. Hà and N. H. V. Hưng, *On the behavior of the algebraic transfer*, *Trans. Amer. Math. Soc.* 357 (2005) 473-487, MR2095619.

- [3] D. P. Carlisle and R. M. W. Wood, *The boundedness conjecture for the action of the Steenrod algebra on polynomials*, in: N. Ray and G. Walker (ed.), Adams Memorial Symposium on Algebraic Topology 2, (Manchester, 1990), in: London Math. Soc. Lecture Notes Ser., Cambridge Univ. Press, Cambridge, vol. 176, 1992, pp. 203-216, MR1232207.
- [4] M. C. Crabb and J. R. Hubbuck, *Representations of the homology of  $BV$  and the Steenrod algebra II*, in: Algebraic Topology: New Trend in Localization and Periodicity, (Sant Feliu de Guíxols, 1994), in: Progr. Math., Birkhäuser Verlag, Basel, Switzerland, vol. 136, 1996, pp. 143-154, MR1397726.
- [5] V. Giambalvo and F. P. Peterson,  *$\mathcal{A}$ -generators for ideals in the Dickson algebra*, J. Pure Appl. Algebra 158 (2001) 161-182, MR1822839.
- [6] L. M. Hà, *Sub-Hopf algebras of the Steenrod algebra and the Singer transfer*, in: Proceedings of the International School and Conference in Algebraic Topology, Hà Nội 2004, Geom. Topol. Monogr., Geom. Topol. Publ., Coventry, vol. 11, 2007, 81-105, MR2402802.
- [7] N. H. V. Hùng, *The cohomology of the Steenrod algebra and representations of the general linear groups*, Trans. Amer. Math. Soc. 357 (2005) 4065-4089, MR2159700.
- [8] N. H. V. Hùng and T. N. Nam, *The hit problem for the Dickson algebra*, Trans. Amer. Math. Soc. 353 (2001) 5029-5040, MR1852092.
- [9] N. H. V. Hùng and T. N. Nam, *The hit problem for the modular invariants of linear groups*, Journal of Algebra 246 (2001) 367-384, MR2872626.
- [10] N. H. V. Hùng and F. P. Peterson,  *$\mathcal{A}$ -generator for the Dickson algebra*, Trans. Amer. Math. Soc., 347 (1995) 4687-4728, MR1316852.
- [11] N. H. V. Hùng and F. P. Peterson, *Spherical classes and the Dickson algebra*, Math. Proc. Camb. Phil. Soc. 124 (1998) 253-264, MR1631123.
- [12] A. S. Janfada and R. M. W. Wood, *The hit problem for symmetric polynomials over the Steenrod algebra*, Math. Proc. Cambridge Philos. Soc. 133 (2002) 295-303, MR1912402.
- [13] A. S. Janfada and R. M. W. Wood, *Generating  $H^*(BO(3), \mathbb{F}_2)$  as a module over the Steenrod algebra*, Math. Proc. Cambridge Philos. Soc. 134 (2003) 239-258, MR1972137.
- [14] M. Kameko, *Products of projective spaces as Steenrod modules*, PhD Thesis, The Johns Hopkins University, ProQuest LLC, Ann Arbor, MI, 1990, 29 pp., MR2638633.
- [15] M. Kameko, *Generators of the cohomology of  $BV_3$* , J. Math. Kyoto Univ. 38 (1998) 587-593, MR1661173.
- [16] M. Kameko, *Generators of the cohomology of  $BV_4$* , Toyama University, Japan, Preprint, 2003, 8 pp.
- [17] N. Minami, *The iterated transfer analogue of the new doomsday conjecture*, Trans. Amer. Math. Soc. 351 (1999) 2325-2351, MR1443884.
- [18] M. F. Mothebe, *Admissible monomials and generating sets for the polynomial algebra as a module over the Steenrod algebra*, Afr. Diaspora J. Math. 16 (2013) 18-27, MR3091712.
- [19] M. F. Mothebe, *Dimension result for the polynomial algebra  $\mathbb{F}_2[x_1, \dots, x_n]$  as a module over the Steenrod algebra*, Int. J. Math. Math. Sci. 2013, Art. ID 150704, 6 pp., MR3144989.
- [20] T. N. Nam,  *$\mathcal{A}$ -générateurs génériques pour l'algèbre polynomiale*, Adv. Math. 186 (2004) 334-362, MR2073910.
- [21] T. N. Nam, *Transfert algébrique et action du groupe linéaire sur les puissances divisées modulo 2*, Ann. Inst. Fourier (Grenoble) 58 (2008) 1785-1837, MR2445834.
- [22] F. P. Peterson, *Generators of  $H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)$  as a module over the Steenrod algebra*, Abstracts Amer. Math. Soc. No. 833 (April 1987).
- [23] F. P. Peterson,  *$\mathcal{A}$ -generators for certain polynomial algebras*, Math. Proc. Cambridge Philos. Soc. 105 (1989) 311-312, MR0974987.
- [24] S. Priddy, *On characterizing summands in the classifying space of a group, I*, Amer. Jour. Math. 112 (1990) 737-748, MR1073007.
- [25] J. Repka and P. Selick, *On the subalgebra of  $H_*((\mathbb{R}P^\infty)^n; \mathbb{F}_2)$  annihilated by Steenrod operations*, J. Pure Appl. Algebra 127 (1998) 273-288, MR1617199.
- [26] J. H. Silverman, *Hit polynomials and the canonical antiautomorphism of the Steenrod algebra*, Proc. Amer. Math. Soc. 123 (1995) 627-637, MR1254854.
- [27] J. H. Silverman and W. M. Singer, *On the action of Steenrod squares on polynomial algebras II*, J. Pure Appl. Algebra 98 (1995) 95-103, MR1317001.
- [28] W. M. Singer, *The transfer in homological algebra*, Math. Zeit. 202 (1989) 493-523, MR1022818.

- [29] W. M. Singer, *On the action of the Steenrod squares on polynomial algebras*, Proc. Amer. Math. Soc. 111 (1991) 577-583, MR1045150.
- [30] N. E. Steenrod and D. B. A. Epstein, *Cohomology operations*, Ann. of Math. Stud. vol. 50, Princeton Univ. Press, Princeton, N.J 1962, MR0145525.
- [31] N. Sum, *The hit problem for the polynomial algebra of four variables*, Quy Nhơn University, Việt Nam, Preprint, 2007, 240 pp., Available online at: <http://arxiv.org/abs/1412.1709>.
- [32] N. Sum, *The negative answer to Kameko's conjecture on the hit problem*, C. R. Acad. Sci. Paris, Ser. I 348 (2010) 669-672, MR2652495.
- [33] N. Sum, *The negative answer to Kameko's conjecture on the hit problem*, Adv. Math. 225 (2010) 2365-2390, MR2680169.
- [34] N. Sum, *On the hit problem for the polynomial algebra*, C. R. Math. Acad. Sci. Paris, Ser. I, 351 (2013) 565-568, MR3095107.
- [35] G. Walker and R. M. W. Wood, *Young tableaux and the Steenrod algebra*, in: Proceedings of the International School and Conference in Algebraic Topology, Hà Nội 2004, Geom. Topol. Monogr., Geom. Topol. Publ., Coventry, vol. 11, 2007, 379-397, MR2402815.
- [36] G. Walker and R. M. W. Wood, *Weyl modules and the mod 2 Steenrod algebra*, J. Algebra 311 (2007) 840-858, MR2314738.
- [37] G. Walker and R. M. W. Wood, *Flag modules and the hit problem for the Steenrod algebra*, Math. Proc. Cambridge Philos. Soc. 147 (2009) 143-171, MR2507313.
- [38] R. M. W. Wood, *Steenrod squares of polynomials and the Peterson conjecture*, Math. Proc. Cambridge Phil. Soc. 105 (1989) 307-309, MR0974986.
- [39] R. M. W. Wood, *Problems in the Steenrod algebra*, Bull. London Math. Soc. 30 (1998) 449-517, MR1643834.
- [40] R. M. W. Wood, *Hit problems and the Steenrod algebra*, in: Proceedings of the summer school "Interactions between algebraic topology and invariant theory", a satellite conference of the third European congress of mathematics, Lecture Course, Ioannina University, Greece, June 2000, Published in Ioannina University reports, 2001, pp. 65-103.

Department of Mathematics, Quy Nhơn University,  
170 An Dương Vương, Quy Nhơn, Bình Định, Việt Nam.

E-mail: nguyensum@qnu.edu.vn

## APPENDIX

In the appendix, we explicitly determine  $(QP_4)_{45}$  by using the algorism presented in the proof of Proposition 3.3. For  $k = 4$ , the degree  $n = 45$  is entry with  $d_1 = 5, d_2 = d_3 = 3$  and  $m = 3$ . It is well known that  $\dim(QP_3)_3 = 7$  and

$$B_3(3) = \{x_3^3, x_2x_3^2, x_2^3, x_1x_3^2, x_1x_2x_3, x_1x_2^2, x_1^3\}.$$

For simplicity, we denote the monomial  $y = x_1^a x_2^b x_3^c \in B_3(3)$  by  $(abc)$  with  $0 \leq a, b, c \leq 3$ . By Theorem 1.3,  $\dim(QP_4)_{45} = (2^4 - 1) \dim(QP_3)_3 = 105$  and  $\Phi(B_3(n))$  is the minimal set of generators for  $\mathcal{A}$ -module  $P_4$  in degree  $n = 45$ . From the proof of Proposition 3.3,  $\Phi(B_3(n)) = \{\phi_{(i;I)}(X^7 y^8) : (i; I) \in \mathcal{N}_4, y \in B_3(3)\}$ . The monomials  $\phi_{(i;I)}(X^7 y^8)$  are determine by the following table.

$(i; I)$	$y$	$\phi_{(i;I)}(X^7 y^8)$	$(i; I)$	$y$	$\phi_{(i;I)}(X^7 y^8)$
$(1; \emptyset)$	(003)	$x_2^7 x_3^7 x_4^{31}$	$(1; \emptyset)$	(012)	$x_2^7 x_3^{15} x_4^{23}$
$(1; \emptyset)$	(030)	$x_2^7 x_3^{31} x_4^7$	$(1; \emptyset)$	(102)	$x_2^{15} x_3^7 x_4^{23}$
$(1; \emptyset)$	(111)	$x_2^{15} x_3^{15} x_4^{15}$	$(1; \emptyset)$	(120)	$x_2^{15} x_3^{23} x_4^7$
$(1; \emptyset)$	(300)	$x_2^{31} x_3^7 x_4^7$	$(1; 2)$	(003)	$x_1 x_2^6 x_3^7 x_4^{31}$
$(1; 2)$	(012)	$x_1 x_2^6 x_3^{15} x_4^{23}$	$(1; 2)$	(030)	$x_1 x_2^6 x_3^{31} x_4^7$
$(1; 2)$	(102)	$x_1 x_2^{14} x_3^7 x_4^{23}$	$(1; 2)$	(111)	$x_1 x_2^{14} x_3^{15} x_4^{15}$
$(1; 2)$	(120)	$x_1 x_2^{14} x_3^{23} x_4^7$	$(1; 2)$	(300)	$x_1 x_2^{30} x_3^7 x_4^7$
$(1; 3)$	(003)	$x_1 x_2^7 x_3^6 x_4^{31}$	$(1; 3)$	(012)	$x_1 x_2^7 x_3^{14} x_4^{23}$
$(1; 3)$	(030)	$x_1 x_2^7 x_3^{30} x_4^7$	$(1; 3)$	(102)	$x_1 x_2^{15} x_3^6 x_4^{23}$
$(1; 3)$	(111)	$x_1 x_2^{15} x_3^{14} x_4^{15}$	$(1; 3)$	(120)	$x_1 x_2^{15} x_3^{22} x_4^7$
$(1; 3)$	(300)	$x_1 x_2^{31} x_3^6 x_4^7$	$(1; 4)$	(003)	$x_1 x_2^7 x_3^7 x_4^{30}$
$(1; 4)$	(012)	$x_1 x_2^7 x_3^{15} x_4^{22}$	$(1; 4)$	(030)	$x_1 x_2^7 x_3^{31} x_4^6$
$(1; 4)$	(102)	$x_1 x_2^{15} x_3^7 x_4^{22}$	$(1; 4)$	(111)	$x_1 x_2^{15} x_3^{15} x_4^{14}$
$(1; 4)$	(120)	$x_1 x_2^{15} x_3^{23} x_4^6$	$(1; 4)$	(300)	$x_1 x_2^{31} x_3^7 x_4^6$
$(2; \emptyset)$	(003)	$x_1^7 x_3^7 x_4^{31}$	$(2; \emptyset)$	(012)	$x_1^7 x_3^{15} x_4^{23}$
$(2; \emptyset)$	(030)	$x_1^7 x_3^{31} x_4^7$	$(2; \emptyset)$	(102)	$x_1^{15} x_3^7 x_4^{23}$
$(2; \emptyset)$	(111)	$x_1^{15} x_3^{15} x_4^{15}$	$(2; \emptyset)$	(120)	$x_1^{15} x_3^{23} x_4^7$
$(2; \emptyset)$	(300)	$x_1^{31} x_3^7 x_4^7$	$(3; \emptyset)$	(003)	$x_1^7 x_2^7 x_4^{31}$
$(3; \emptyset)$	(012)	$x_1^7 x_2^{15} x_4^{23}$	$(3; \emptyset)$	(030)	$x_1^7 x_2^{31} x_4^7$
$(3; \emptyset)$	(102)	$x_1^{15} x_2^7 x_4^{23}$	$(3; \emptyset)$	(111)	$x_1^{15} x_2^{15} x_4^{15}$
$(3; \emptyset)$	(120)	$x_1^{15} x_2^{23} x_4^7$	$(3; \emptyset)$	(300)	$x_1^{31} x_2^7 x_4^7$
$(2; 3)$	(003)	$x_1^7 x_2 x_3^6 x_4^{31}$	$(2; 3)$	(012)	$x_1^7 x_2 x_3^{14} x_4^{23}$
$(2; 3)$	(030)	$x_1^7 x_2 x_3^{30} x_4^7$	$(2; 3)$	(102)	$x_1^{15} x_2 x_3^6 x_4^{23}$
$(2; 3)$	(111)	$x_1^{15} x_2 x_3^{14} x_4^{15}$	$(2; 3)$	(120)	$x_1^{15} x_2 x_3^6 x_4^{23}$
$(2; 3)$	(300)	$x_1^{31} x_2 x_3^6 x_4^7$	$(2; 4)$	(003)	$x_1^7 x_2 x_3^7 x_4^{30}$
$(2; 4)$	(012)	$x_1^7 x_2 x_3^{14} x_4^{23}$	$(2; 4)$	(030)	$x_1^7 x_2 x_3^{31} x_4^6$



$(i; I)$	$y$	$\phi_{(i; I)}(X^7 y^8)$	$(i; I)$	$y$	$\phi_{(i; I)}(X^7 y^8)$
(2; 4)	(102)	$x_1^{15} x_2 x_3^7 x_4^{22}$	(2; 4)	(111)	$x_1^{15} x_2 x_3^{15} x_4^{14}$
(2; 4)	(120)	$x_1^{15} x_2 x_3^{23} x_4^{14}$	(2; 4)	(300)	$x_1^{31} x_2 x_3^7 x_4^6$
(3; 4)	(003)	$x_1^7 x_2^7 x_3 x_4^{30}$	(3; 4)	(012)	$x_1^7 x_2^{15} x_3 x_4^{22}$
(3; 4)	(030)	$x_1^7 x_2^{31} x_3 x_4^6$	(3; 4)	(102)	$x_1^{15} x_2^7 x_3 x_4^{14}$
(3; 4)	(111)	$x_1^{15} x_2^{15} x_3 x_4^{14}$	(3; 4)	(120)	$x_1^{15} x_2^{23} x_3 x_4^6$
(3; 4)	(300)	$x_1^{31} x_2^7 x_3 x_4^6$	(1; 2, 3)	(003)	$x_1^3 x_2^5 x_3^6 x_4^{31}$
(1; 2, 3)	(012)	$x_1^3 x_2^5 x_3^{14} x_4^{23}$	(1; 2, 3)	(030)	$x_1^3 x_2^5 x_3^{30} x_4^7$
(1; 2, 3)	(102)	$x_1^3 x_2^{13} x_3^6 x_4^{23}$	(1; 2, 3)	(111)	$x_1^3 x_2^{13} x_3^{14} x_4^{15}$
(1; 2, 3)	(120)	$x_1^3 x_2^{13} x_3^{22} x_4^7$	(1; 2, 3)	(300)	$x_1^3 x_2^{29} x_3^6 x_4^7$
(1; 2, 4)	(003)	$x_1^3 x_2^5 x_3^7 x_4^{30}$	(1; 2, 4)	(012)	$x_1^3 x_2^5 x_3^{15} x_4^{22}$
(1; 2, 4)	(030)	$x_1^3 x_2^5 x_3^{31} x_4^6$	(1; 2, 4)	(102)	$x_1^3 x_2^{13} x_3^7 x_4^{22}$
(1; 2, 4)	(111)	$x_1^3 x_2^{13} x_3^{15} x_4^{14}$	(1; 2, 4)	(120)	$x_1^3 x_2^{13} x_3^{23} x_4^6$
(1; 2, 4)	(300)	$x_1^3 x_2^{29} x_3^7 x_4^6$	(1; 3, 4)	(003)	$x_1^3 x_2^7 x_3^5 x_4^{30}$
(1; 3, 4)	(012)	$x_1^3 x_2^7 x_3^{13} x_4^{22}$	(1; 3, 4)	(030)	$x_1^3 x_2^7 x_3^{29} x_4^6$
(1; 3, 4)	(102)	$x_1^3 x_2^{15} x_3^5 x_4^{22}$	(1; 3, 4)	(111)	$x_1^3 x_2^{15} x_3^{13} x_4^{14}$
(1; 3, 4)	(120)	$x_1^3 x_2^{15} x_3^{21} x_4^6$	(1; 3, 4)	(300)	$x_1^3 x_2^{31} x_3^5 x_4^6$
(2; 3, 4)	(003)	$x_1^7 x_2^3 x_3^5 x_4^{30}$	(2; 3, 4)	(012)	$x_1^7 x_2^3 x_3^{13} x_4^{22}$
(2; 3, 4)	(030)	$x_1^7 x_2^3 x_3^{29} x_4^6$	(2; 3, 4)	(102)	$x_1^{15} x_2^3 x_3^5 x_4^{22}$
(2; 3, 4)	(111)	$x_1^{15} x_2^3 x_3^{13} x_4^{14}$	(2; 3, 4)	(120)	$x_1^{15} x_2^3 x_3^{21} x_4^6$
(2; 3, 4)	(300)	$x_1^{31} x_2^3 x_3^5 x_4^6$	(4; $\emptyset$ )	(003)	$x_1^7 x_2^7 x_3^{31}$
(4; $\emptyset$ )	(012)	$x_1^7 x_2^{15} x_3^{23}$	(4; $\emptyset$ )	(030)	$x_1^7 x_2^{31} x_3^7$
(4; $\emptyset$ )	(102)	$x_1^{15} x_2^7 x_3^{23}$	(4; $\emptyset$ )	(111)	$x_1^{15} x_2^{15} x_3^{15}$
(4; $\emptyset$ )	(120)	$x_1^{15} x_2^{23} x_3^7$	(4; $\emptyset$ )	(300)	$x_1^{31} x_2^7 x_3^7$
(1; $I_1$ )	(003)	$x_1^7 x_2^7 x_3^7 x_4^{24}$	(1; $I_1$ )	(012)	$x_1^7 x_2^7 x_3^9 x_4^{22}$
(1; $I_1$ )	(030)	$x_1^7 x_2^7 x_3^{25} x_4^6$	(1; $I_1$ )	(102)	$x_1^7 x_2^{11} x_3^5 x_4^{22}$
(1; $I_1$ )	(111)	$x_1^7 x_2^{11} x_3^{13} x_4^{14}$	(1; $I_1$ )	(120)	$x_1^7 x_2^{11} x_3^{21} x_4^6$
(1; $I_1$ )	(300)	$x_1^7 x_2^{27} x_3^5 x_4^6$			

Now, we compute  $\phi_{(i; I)}(X^7 \bar{y}^8)$  in terms of the monomials in  $B_4(45) = \Phi(B_3(45))$  with  $\bar{y}$  a monomial in  $B_4(3)$  such that  $\nu_i(\bar{y}) > 0$  and  $(i; I) \in \mathcal{N}_4$ . It is easy to see that  $B_4(3) = \Phi^0(B_3(3))$  is the set consisting of all the following monomials

$$x_4^3, x_3 x_4^2, x_3^3, x_2 x_4^2, x_2 x_3 x_4, x_2 x_3^2, x_2^3, x_1 x_4^2, \\ x_1 x_3 x_4, x_1 x_3^2, x_1 x_2 x_4, x_1 x_2 x_3, x_1 x_2^2, x_1^3.$$

In the following table, we denote the monomial  $\bar{y} = x_1^a x_2^b x_3^c x_4^d$  by  $(abcd)$  with  $0 \leq a, b, c, d \leq 3$ . If the monomial  $\bar{y}$  satisfies the conditions of Case 3.1.u in the proof of Proposition 3.3, then we denote  $\phi_{(i; I)}(X^7 \bar{y}^8)$  by  $\phi_{(i; I)}^{(u)} \bar{y}^8$ . Here  $1 \leq u \leq 14$ .

$(i; I)$	$\bar{y}$	Case	$\phi_{(i;I)}(X^7)\bar{y}^8 \equiv$
$(1; \emptyset)$	(1002)	4	$x_1x_2^{14}x_3^7x_4^{23} + x_1x_2^7x_3^{14}x_4^{23} + x_1x_2^7x_3^7x_4^{30}$
$(1; \emptyset)$	(1011)	4	$x_1x_2^{14}x_3^{15}x_4^{15} + x_1x_2^7x_3^{14}x_4^{23} + x_1x_2^7x_3^{15}x_4^{22}$
$(1; \emptyset)$	(1020)	4	$x_1x_2^{14}x_3^{23}x_4^7 + x_1x_2^7x_3^{30}x_4^7 + x_1x_2^7x_3^{15}x_4^{22}$
$(1; \emptyset)$	(1101)	4	$x_1x_2^{14}x_3^7x_4^{23} + x_1x_2^{15}x_3^{14}x_4^{15} + x_1x_2^{15}x_3^7x_4^{22}$
$(1; \emptyset)$	(1110)	4	$x_1x_2^{14}x_3^{23}x_4^7 + x_1x_2^{15}x_3^{22}x_4^7 + x_1x_2^{15}x_3^{15}x_4^{14}$
$(1; \emptyset)$	(1200)	4	$x_1x_2^{30}x_3^7x_4^7 + x_1x_2^{15}x_3^{22}x_4^7 + x_1x_2^{15}x_3^7x_4^{22}$
$(1; 2)$	(1002)	4	$x_1x_2^{14}x_3^7x_4^{23} + x_1^3x_2^5x_3^{14}x_4^{23} + x_1^3x_2^5x_3^7x_4^{30}$
$(1; 2)$	(1011)	4	$x_1x_2^{14}x_3^{15}x_4^{15} + x_1^3x_2^5x_3^{14}x_4^{23} + x_1^3x_2^5x_3^{15}x_4^{22}$
$(1; 2)$	(1020)	4	$x_1x_2^{14}x_3^{23}x_4^7 + x_1^3x_2^5x_3^{30}x_4^7 + x_1^3x_2^5x_3^{15}x_4^{22}$
$(1; 2)$	(1101)	4	$x_1x_2^{14}x_3^7x_4^{23} + x_1^3x_2^{13}x_3^{14}x_4^{15} + x_1^3x_2^{13}x_3^7x_4^{22}$
$(1; 2)$	(1110)	4	$x_1x_2^{14}x_3^{23}x_4^7 + x_1^3x_2^{13}x_3^{22}x_4^7 + x_1^3x_2^{13}x_3^{15}x_4^{14}$
$(1; 2)$	(1200)	4	$x_1x_2^{30}x_3^7x_4^7 + x_1^3x_2^{13}x_3^{22}x_4^7 + x_1^3x_2^{13}x_3^7x_4^{22}$
$(1; 3)$	(1002)	4	$x_1^3x_2^{13}x_3^6x_4^{23} + x_1x_2^7x_3^{14}x_4^{23} + x_1^3x_2^5x_3^7x_4^{30}$
$(1; 3)$	(1011)	4	$x_1^3x_2^{13}x_3^{14}x_4^{15} + x_1x_2^7x_3^{14}x_4^{23} + x_1^3x_2^7x_3^{13}x_4^{22}$
$(1; 3)$	(1020)	4	$x_1^3x_2^{13}x_3^{22}x_4^7 + x_1x_2^7x_3^{30}x_4^7 + x_1^3x_2^7x_3^{13}x_4^{22}$
$(1; 3)$	(1101)	4	$x_1^3x_2^{13}x_3^6x_4^{23} + x_1x_2^{15}x_3^{14}x_4^{15} + x_1^3x_2^{15}x_3^5x_4^{22}$
$(1; 3)$	(1110)	4	$x_1^3x_2^{13}x_3^{22}x_4^7 + x_1x_2^{15}x_3^{22}x_4^7 + x_1^3x_2^{15}x_3^{13}x_4^{14}$
$(1; 3)$	(1200)	4	$x_1^3x_2^{29}x_3^6x_4^7 + x_1x_2^{15}x_3^{22}x_4^7 + x_1^3x_2^{15}x_3^5x_4^{22}$
$(1; 4)$	(1002)	4	$x_1^3x_2^{13}x_3^7x_4^{22} + x_1^3x_2^7x_3^{13}x_4^{22} + x_1x_2^7x_3^7x_4^{30}$
$(1; 4)$	(1011)	4	$x_1^3x_2^{13}x_3^{15}x_4^{14} + x_1^3x_2^7x_3^{13}x_4^{22} + x_1x_2^7x_3^{15}x_4^{22}$
$(1; 4)$	(1020)	4	$x_1^3x_2^{13}x_3^{23}x_4^6 + x_1^3x_2^7x_3^{29}x_4^6 + x_1x_2^7x_3^{15}x_4^{22}$
$(1; 4)$	(1101)	4	$x_1^3x_2^{13}x_3^7x_4^{22} + x_1^3x_2^{15}x_3^{13}x_4^{14} + x_1x_2^{15}x_3^7x_4^{22}$
$(1; 4)$	(1110)	4	$x_1^3x_2^{13}x_3^{23}x_4^6 + x_1^3x_2^{15}x_3^{21}x_4^6 + x_1x_2^{15}x_3^{15}x_4^{14}$
$(1; 4)$	(1200)	4	$x_1^3x_2^{29}x_3^6x_4^6 + x_1^3x_2^{15}x_3^{21}x_4^6 + x_1x_2^{15}x_3^7x_4^{22}$
$(2; \emptyset)$	(0102)	4	$\phi_{(1;2)}^{(4)}(1002)^8 + x_1^7x_2x_3^{14}x_4^{23} + x_1^7x_2x_3^7x_4^{30}$
$(2; \emptyset)$	(0111)	4	$\phi_{(1;2)}^{(4)}(1011)^8 + x_1^7x_2x_3^{14}x_4^{23} + x_1^7x_2x_3^{15}x_4^{22}$
$(2; \emptyset)$	(0120)	4	$\phi_{(1;2)}^{(4)}(1020)^8 + x_1^7x_2x_3^{30}x_4^7 + x_1^7x_2x_3^{15}x_4^{22}$
$(3; \emptyset)$	(0012)	5	$\phi_{(1;\emptyset)}^{(4)}(1002)^8 + \phi_{(2;\emptyset)}^{(4)}(0102)^8 + x_1^7x_2^7x_3^7x_4^{24}$
$(4; \emptyset)$	(0012)	6	$\phi_{(1;\emptyset)}^{(4)}(1011)^8 + \phi_{(2;\emptyset)}^{(4)}(0111)^8 + \phi_{(3;\emptyset)}^{(5)}(0012)^8$
$(3; \emptyset)$	(0030)	7	$\phi_{(1;3)}^{(4)}(1020)^8 + \phi_{(2;3)}^{(4)}(0120)^8 + x_1^7x_2^7x_3^9x_4^{22}$
$(2; 3)$	(0102)	8	$\phi_{(1;3)}^{(4)}(1002)^8 + \phi_{(3;\emptyset)}^{(5)}(0012)^8 + x_1^7x_2^7x_3^9x_4^{22}$
$(2; 3)$	(0111)	8	$\phi_{(1;3)}^{(4)}(1011)^8 + \phi_{(3;\emptyset)}^{(5)}(0012)^8 + x_1^7x_2^7x_3^9x_4^{22}$
$(2; 3)$	(0120)	8	$\phi_{(1;3)}^{(4)}(1020)^8 + \phi_{(3;\emptyset)}^{(7)}(0030)^8 + x_1^7x_2^7x_3^9x_4^{22}$
$(2; 4)$	(0102)	8	$\phi_{(1;4)}^{(4)}(1002)^8 + x_1^7x_2^7x_3^9x_4^{22} + x_1^7x_2^7x_3^7x_4^{24}$
$(2; 4)$	(0111)	8	$\phi_{(1;4)}^{(4)}(1011)^8 + \phi_{(4;\emptyset)}^{(6)}(0012)^8 + x_1^7x_2^7x_3^9x_4^{22}$
$(2; 4)$	(0120)	8	$\phi_{(1;4)}^{(4)}(1020)^8 + \phi_{(4;\emptyset)}^{(6)}(0012)^8 + x_1^7x_2^7x_3^{25}x_4^6$

$(i; I)$	$\bar{y}$	Case	$\phi_{(i;I)}(X^7)\bar{y}^8 \equiv$
$(3; \emptyset)$	$(0111)$	9	$\phi_{(1;3)}^{(4)}(1101)^8 + \phi_{(2;3)}^{(8)}(0102)^8 + x_1^7 x_2^{15} x_3 x_4^{22}$
$(3; 4)$	$(0111)$	9	$\phi_{(1;4)}^{(4)}(1101)^8 + \phi_{(2;4)}^{(8)}(0102)^8 + \phi_{(1;\emptyset)}^{(4)}(1101)^8$ $+ \phi_{(2;\emptyset)}^{(4)}(0102)^8 + \phi_{(3;\emptyset)}^{(9)}(0111)^8$
$(3; \emptyset)$	$(0120)$	9	$\phi_{(1;3)}^{(4)}(1110)^8 + \phi_{(2;3)}^{(8)}(0120)^8 + \phi_{(3;4)}^{(9)}(0111)^8$
$(3; 4)$	$(0120)$	9	$\phi_{(1;4)}^{(4)}(1110)^8 + \phi_{(2;4)}^{(8)}(0120)^8 + \phi_{(1;\emptyset)}^{(4)}(1110)^8$ $+ \phi_{(2;\emptyset)}^{(4)}(0120)^8 + \phi_{(3;\emptyset)}^{(9)}(0120)^8$
$(2; \emptyset)$	$(0300)$	10	$\phi_{(1;2)}^{(4)}(1020)^8 + \phi_{(2;3)}^{(8)}(0102)^8 + \phi_{(2;4)}^{(8)}(0102)^8$
$(2; 3)$	$(0300)$	10	$\phi_{(1;3)}^{(4)}(1200)^8 + \phi_{(3;\emptyset)}^{(9)}(0120)^8 + x_1^7 x_2^{15} x_3 x_4^{22}$
$(2; 4)$	$(0300)$	10	$\phi_{(1;4)}^{(4)}(1200)^8 + \phi_{(3;4)}^{(9)}(0120)^8 + \phi_{(1;\emptyset)}^{(4)}(1200)^8$ $+ \phi_{(2;\emptyset)}^{(10)}(0300)^8 + \phi_{(3;\emptyset)}^{(9)}(0120)^8$
$(1; 2, 3)$	$(1002)$	11	$\phi_{(2;3)}^{(8)}(0102)^8 + x_1^7 x_2 x_3^{14} x_4^{23} + x_1^7 x_2^3 x_3^5 x_4^{30}$
$(1; 2, 3)$	$(1011)$	11	$\phi_{(2;3)}^{(8)}(0111)^8 + x_1^7 x_2 x_3^{14} x_4^{23} + x_1^7 x_2^3 x_3^{13} x_4^{22}$
$(1; 2, 3)$	$(1020)$	11	$\phi_{(2;3)}^{(8)}(0120)^8 + x_1^7 x_2 x_3^{30} x_4^7 + x_1^7 x_2^3 x_3^{13} x_4^{22}$
$(1; 2, 3)$	$(1101)$	11	$\phi_{(2;3)}^{(8)}(0102)^8 + \phi_{(2;3)}^{(8)}(0111)^8 + x_1^7 x_2^{11} x_3^5 x_4^{22}$
$(1; 2, 3)$	$(1110)$	11	$x_1^7 x_2^{11} x_3^{13} x_4^{14}$
$(1; 2, 3)$	$(1200)$	11	$\phi_{(2;3)}^{(8)}(0300)^8 + \phi_{(2;3)}^{(8)}(0120)^8 + x_1^7 x_2^{11} x_3^5 x_4^{22}$
$(1; 2, 4)$	$(1002)$	11	$\phi_{(2;4)}^{(8)}(0102)^8 + x_1^7 x_2^3 x_3^{13} x_4^{22} + x_1^7 x_2 x_3^7 x_4^{30}$
$(1; 2, 4)$	$(1011)$	11	$\phi_{(2;4)}^{(8)}(0111)^8 + x_1^7 x_2^3 x_3^{13} x_4^{22} + x_1^7 x_2 x_3^{15} x_4^{22}$
$(1; 2, 4)$	$(1020)$	11	$\phi_{(2;4)}^{(8)}(0120)^8 + x_1^7 x_2^3 x_3^{29} x_4^6 + x_1^7 x_2 x_3^{15} x_4^{22}$
$(1; 2, 4)$	$(1101)$	11	$x_1^7 x_2^{11} x_3^{13} x_4^{14}$
$(1; 2, 4)$	$(1110)$	11	$\phi_{(2;4)}^{(8)}(0120)^8 + \phi_{(2;4)}^{(8)}(0111)^8 + x_1^7 x_2^{11} x_3^{21} x_4^6$
$(1; 2, 4)$	$(1200)$	11	$\phi_{(2;4)}^{(10)}(0300)^8 + \phi_{(2;4)}^{(8)}(0102)^8 + x_1^7 x_2^{11} x_3^{21} x_4^6$
$(1; 3, 4)$	$(1002)$	11	$x_1^7 x_2^{11} x_3^5 x_4^{22} + x_1^7 x_2^7 x_3^9 x_4^{22} + x_1^7 x_2^7 x_3 x_4^{30}$
$(1; 3, 4)$	$(1011)$	11	$x_1^7 x_2^{11} x_3^{13} x_4^{14}$
$(1; 3, 4)$	$(1020)$	11	$x_1^7 x_2^{11} x_3^{21} x_4^6 + x_1^7 x_2^7 x_3^{25} x_4^6 + x_1^7 x_2^7 x_3^9 x_4^{22}$
$(1; 3, 4)$	$(1101)$	11	$\phi_{(3;4)}^{(9)}(0111)^8 + x_1^7 x_2^{11} x_3^5 x_4^{22} + x_1^7 x_2^{15} x_3 x_4^{22}$
$(1; 3, 4)$	$(1110)$	11	$\phi_{(3;4)}^{(9)}(0120)^8 + \phi_{(3;4)}^{(9)}(0111)^8 + x_1^7 x_2^{11} x_3^{21} x_4^6$
$(1; 3, 4)$	$(1200)$	11	$\phi_{(3;4)}^{(9)}(0120)^8 + x_1^7 x_2^{27} x_3^5 x_4^6 + x_1^7 x_2^{15} x_3 x_4^{22}$
$(2; \emptyset)$	$(1101)$	12	$\phi_{(1;2)}^{(4)}(1020)^8 + x_1^{15} x_2 x_3^{14} x_4^{15} + x_1^{15} x_2 x_3^7 x_4^{22}$
$(2; \emptyset)$	$(1110)$	12	$\phi_{(1;2)}^{(4)}(1020)^8 + x_1^{15} x_2 x_3^{22} x_4^7 + x_1^{15} x_2 x_3^{15} x_4^{14}$
$(2; 3)$	$(1101)$	12	$\phi_{(1;2,3)}^{(11)}(1002)^8 + x_1^{15} x_2 x_3^{14} x_4^{15} + x_1^{15} x_2^3 x_3^5 x_4^{22}$
$(2; 3)$	$(1110)$	12	$\phi_{(1;2,3)}^{(11)}(1020)^8 + x_1^{15} x_2 x_3^{22} x_4^7 + x_1^{15} x_2^3 x_3^{13} x_4^{14}$
$(2; 4)$	$(1101)$	12	$\phi_{(1;2,4)}^{(11)}(1002)^8 + x_1^{15} x_2^3 x_3^{13} x_4^{14} + x_1^{15} x_2 x_3^7 x_4^{22}$
$(2; 4)$	$(1110)$	12	$\phi_{(1;2,4)}^{(11)}(1020)^8 + x_1^{15} x_2^3 x_3^{21} x_4^6 + x_1^{15} x_2 x_3^{15} x_4^{14}$

$(i; I)$	$\bar{y}$	Case	$\phi_{(i;I)}(X^7)\bar{y}^8 \equiv$
(2; 3, 4)	(1101)	12	$\phi_{(1;3,4)}^{(11)}(1002)^8 + \phi_{(1;\emptyset)}^{(4)}(1002)^8 + \phi_{(1;3)}^{(4)}(1002)^8$ $+ \phi_{(1;4)}^{(4)}(1002)^8 + \phi_{(2;\emptyset)}^{(12)}(1101)^8$ $+ \phi_{(2;3)}^{(12)}(1101)^8 + \phi_{(2;4)}^{(12)}(1101)^8$
(2; 3, 4)	(1110)	12	$\phi_{(1;3,4)}^{(11)}(1020)^8 + \phi_{(1;\emptyset)}^{(4)}(1020)^8 + \phi_{(1;3)}^{(4)}(1020)^8$ $+ \phi_{(1;4)}^{(4)}(1020)^8 + \phi_{(2;\emptyset)}^{(12)}(1110)^8$ $+ \phi_{(2;3)}^{(12)}(1110)^8 + \phi_{(2;4)}^{(12)}(1110)^8$
(2; $\emptyset$ )	(1200)	12	$\phi_{(1;2)}^{(4)}(1200)^8 + \phi_{(2;3)}^{(12)}(1110)^8 + \phi_{(2;4)}^{(12)}(1101)^8$
(2; 3)	(1200)	12	$\phi_{(1;2,3)}^{(11)}(1200)^8 + \phi_{(2;3)}^{(12)}(1110)^8 + \phi_{(2;3,4)}^{(12)}(1101)^8$
(2; 4)	(1200)	12	$\phi_{(1;2,4)}^{(11)}(1200)^8 + \phi_{(2;3,4)}^{(12)}(1110)^8 + \phi_{(2;4)}^{(12)}(1101)^8$
(2; 3, 4)	(1200)	12	$\phi_{(1;3,4)}^{(11)}(1200)^8 + \phi_{(1;\emptyset)}^{(4)}(1200)^8 + \phi_{(1;3)}^{(4)}(1200)^8$ $+ \phi_{(1;4)}^{(4)}(1200)^8 + \phi_{(2;\emptyset)}^{(12)}(1200)^8$ $+ \phi_{(2;3)}^{(12)}(1200)^8 + \phi_{(2;4)}^{(12)}(1200)^8$
(1; $\emptyset$ )	(3000)	13	$\phi_{(1;2)}^{(4)}(1200)^8 + \phi_{(1;3)}^{(4)}(1020)^8 + \phi_{(1;4)}^{(4)}(1002)^8$
(1; 2)	(3000)	13	$\phi_{(1;2)}^{(4)}(1200)^8 + \phi_{(1;2,3)}^{(11)}(1020)^8 + \phi_{(1;2,4)}^{(11)}(1002)^8$
(1; 3)	(3000)	13	$\phi_{(1;2,3)}^{(11)}(1200)^8 + \phi_{(1;3)}^{(4)}(1020)^8 + \phi_{(1;3,4)}^{(11)}(1002)^8$
(1; 4)	(3000)	13	$\phi_{(1;2,4)}^{(11)}(1200)^8 + \phi_{(1;3,4)}^{(11)}(1020)^8 + \phi_{(1;4)}^{(4)}(1002)^8$
(1; 2, 3)	(3000)	13	$\phi_{(2;3)}^{(12)}(1200)^8 + x_1^{15}x_2x_3^{14}x_4^{15} + x_1^{15}x_2^3x_3^5x_4^{22}$
(1; 2, 4)	(3000)	13	$\phi_{(2;4)}^{(12)}(1200)^8 + x_1^{15}x_2^3x_3^{21}x_4^6 + x_1^{15}x_2x_3^7x_4^{22}$
(1; 3, 4)	(3000)	13	$\phi_{(2;3,4)}^{(12)}(1200)^8 + \phi_{(1;\emptyset)}^{(4)}(3000)^8 + \phi_{(1;3)}^{(4)}(3000)^8$ $+ \phi_{(1;4)}^{(4)}(3000)^8 + \phi_{(2;\emptyset)}^{(12)}(1200)^8$ $+ \phi_{(2;3)}^{(12)}(1200)^8 + \phi_{(2;4)}^{(12)}(1200)^8$
(3; $\emptyset$ )	(1011)	14	$\phi_{(1;3)}^{(4)}(1002)^8 + \phi_{(2;3)}^{(12)}(1101)^8 + x_1^{15}x_2^7x_3x_4^{22}$
(3; $\emptyset$ )	(1110)	14	$\phi_{(1;3)}^{(4)}(1200)^8 + \phi_{(2;3)}^{(12)}(1200)^8 + x_1^{15}x_2^{15}x_3x_4^{14}$
(3; 4)	(1011)	14	$\phi_{(1;3,4)}^{(11)}(1002)^8 + \phi_{(2;3,4)}^{(12)}(1101)^8 + x_1^{15}x_2^7x_3x_4^{22}$
(3; 4)	(1020)	14	$\phi_{(1;3,4)}^{(11)}(1020)^8 + \phi_{(2;3,4)}^{(12)}(1110)^8 + \phi_{(3;4)}^{(14)}(1011)^8$
(3; 4)	(1110)	14	$\phi_{(1;3,4)}^{(11)}(1200)^8 + \phi_{(2;3,4)}^{(12)}(1200)^8 + x_1^{15}x_2^{15}x_3x_4^{14}$
(4; $\emptyset$ )	(0111)	14	$\phi_{(1;\emptyset)}^{(4)}(1110)^8 + \phi_{(2;\emptyset)}^{(4)}(0120)^8 + \phi_{(3;\emptyset)}^{(9)}(0120)^8$
(4; $\emptyset$ )	(1101)	14	$\phi_{(1;4)}^{(4)}(1200)^8 + \phi_{(2;4)}^{(12)}(1200)^8 + \phi_{(3;4)}^{(14)}(1110)^8$
(3; $\emptyset$ )	(1020)	14	$\phi_{(1;3)}^{(4)}(1020)^8 + \phi_{(2;3)}^{(12)}(1110)^8 + \phi_{(3;4)}^{(14)}(1011)^8$
(4; $\emptyset$ )	(0102)	14	$\phi_{(1;4)}^{(4)}(1101)^8 + \phi_{(2;4)}^{(8)}(0102)^8 + \phi_{(3;4)}^{(9)}(0111)^8$
(4; $\emptyset$ )	(1002)	14	$\phi_{(1;4)}^{(4)}(1002)^8 + \phi_{(2;4)}^{(12)}(1101)^8 + \phi_{(3;4)}^{(14)}(1011)^8$
(4; $\emptyset$ )	(1011)	14	$\phi_{(1;4)}^{(4)}(1020)^8 + \phi_{(2;4)}^{(12)}(1110)^8 + \phi_{(3;4)}^{(14)}(1020)^8$